THE INVERSE OF MEUSNIER'S THEOREM.

BY PROFESSOR EDWARD KASNER.

(Read before the American Mathematical Society, April 25, 1908.)

MEUSNIER'S theorem, relating to the curves drawn on an arbitrary surface

$$f(x, y, z) = 0, \tag{I}$$

was extended by Lie* to the curves satisfying any Monge equation of the first order

$$f(x, y, z, y', z') = 0, \tag{II}$$

where primes denote differentiation with respect to x. In this note we show that the theorem is valid in the more general case of curves defined by any equation of the form

$$Ay'' + Bz'' + C = 0, \tag{III}$$

where A, B, C are arbitrary functions of $x, y, z, y' z'$; and that no further extension is possible.

Our problem is to find the most general system of space curves with the *Meusnier property*. This deals with the curvature of the curves of the system which pass through a common point O, in a common direction, and may be stated in any one of the following equivalent ways:

1. The radius of curvature varies as the sine of the angle between the osculating plane and a fixed plane.
2. The circles of curvature generate a sphere.
3. The locus of the centers of curvature is a circle through the point O.
4. The locus of the inverse centers of curvature is a straight line.

It will be convenient to use the last statement, sometimes referred to as Hachette's theorem. Taking the origin at the given point O, and the axis of X along the given direction, we find for the center of curvature

* *Leipziger Berichte*, vol. 50 (1898), p. 1; *Math. Annalen*, vol. 59 (1904), p. 299.

$$(4) \qquad X_1 = 0, \quad Y_1 = \frac{y''}{y''^2 + z''^2}, \quad Z_1 = \frac{z''}{y''^2 + z''^2}.$$

The inverse center of curvature, *i. e.*, the inverse of this point with respect to a unit sphere about O, is

$$(5) \qquad X = 0, \quad Y = y'', \quad Z = z''.$$

The Meusnier property requires that the point X, Y, Z, for the curves considered, shall be confined to a straight line. Let this line, which is necessarily in the plane normal to the given lineal element, be

$$(6) \qquad X = 0, \quad AY + BZ + C = 0.$$

Then, from (5), the relation between y'' and z'' must be of the form

$$(\text{III}) \qquad Ay'' + Bz'' + C = 0.$$

The line (6) is fixed for a given lineal element, but may vary when the element varies. Hence A, B, C are arbitrary functions of x, y, z, y', z'.

THEOREM. *The system of curves defined by any differential equation of the form (III) possesses the Meusnier property, i. e., the locus of the inverse centers of curvature of those curves of the system which have a lineal element in common is a straight line. Conversely, every system possessing this property satisfies an equation of the form (III).*

Equations of the types I and II give rise by differentiation to special cases of the type III. More generally we may start with any two-parameter family of surfaces

$$(7) \qquad f(x, y, z, a, b) = 0,$$

or any one-parameter family of Monge equations of the first order

$$(8) \qquad f(x, y, z, y', z', a) = 0,$$

and derive equations of the form III. The systems of curves defined by (7) and (8) constitute special species (the second including the first) of the general class III.

The Meusnier property, although stated in metric form, really belongs to the general geometry whose group is composed of all

(analytic) point transformations. This follows from the fact that in any point transformation the new values of y'' and z'' are linear and integral in the original values. The type III is, therefore, invariant. The species (7) and (8) are also invariant under this group.

The case where the absolute term C vanishes, so that the equation III reduces to

$$Ay'' + Bz'' = 0, \tag{9}$$

is metrically exceptional. The statement 1. of the Meusnier property then becomes illusory since all curves having an element in common here have a common osculating plane. The sphere of statement 2. reduces to this plane, and the loci of 3. and 4. reduce to the same straight line passing through O. The type (9) gives the most general system of space curves such that curves having an element in common also have a common osculating plane. This type is not invariant under the group of all point transformations but is invariant under the projective group.

In connection with the general type III we may develope a theory of *straightest curves* which includes as special cases the theory of geodesics on a surface and the Hertzian theory of path curves in the non-holonomic motion defined by a Pfaffian equation. A curve of the system represented by an equation III is here termed a straightest curve of the system when its curvature at any point is less than (or at most equal to) the curvature of the other curves of the system which pass through that point in the same direction.

The differential equations of these curves can easily be derived by writing down the conditions that the curvature, which is a known function of x, y, z, y', z', here regarded as fixed, and of y'', z'' regarded as variable, but subject to the relation III, shall be a minimum. We prefer, however, to derive the equations from the fact that the locus of the inverse centers of curvatures is a straight line. When the lineal element has its direction given by any values of y' and z', the associated straight line is found to be

$$\begin{gathered} X + y'Y + z'Z = 0, \\ (Ay' + Bz')X - AY - BZ - \frac{C}{1 + y'^2 + z'^2} = 0. \end{gathered} \tag{10}$$

For a curve of least curvature, the osculating plane, whose general equation is

$$\begin{vmatrix} X & Y & Z \\ 1 & y' & z' \\ 0 & y'' & z'' \end{vmatrix} = 0, \tag{11}$$

must meet the line (10) in the point nearest to O. This means that (11) must be perpendicular to (10), so that

$$\begin{vmatrix} Ay' + Bz' & -A & -B \\ 1 & y' & z' \\ 0 & y'' & z'' \end{vmatrix} = 0. \tag{12}$$

Combining this with III, we find, as our final equations,

$$y'' = \frac{-C(A + Ay'^2 + By'z')}{A(A + Ay'^2 + By'z') + B(B + Bz'^2 + Ay'z')},$$
$$z'' = \frac{-C(B + Bz'^2 + Ay'z')}{A(A + Ay'^2 + By'z') + B(B + Bz'^2 + Ay'z')}. \tag{13}$$

Our result may be stated as follows:

*In any system of curves of type (III) there exists a four-fold infinitude of straightest curves, i. e., curves of least curvature, one passing through each point in each direction. The differential equations of these curves are given by (13).**

The systems (13) which thus present themselves are of perfectly general character. For, given any system of ∞^4 space curves, one through each lineal element, its differential equations may be written

$$y'' = g(x, y, z, y', z'), \quad z'' = h(x, y, z, y', z'). \tag{14}$$

By comparing these with (13), the ratios $A : B : C$, and hence an equation of type III, may be found. The inverse problem which thus arises (determination of a system III from its straightest lines) is, in general, capable of only one solution.†

* In the case of type I there are ∞^2 such curves, namely the geodesics. In type II there are ∞^3 straightest curves.

† An exception arises when the assigned system (14) consists of the straight lines $y'' = 0$, $z'' = 0$. The system III may then be any one of the form (9).

It is of interest to notice that the type III presented itself first in connection with the general transformation theory * of the lineal elements of space. Such a transformation is defined by setting $x_1, y_1, z_1, y_1', z_1'$ equal to arbitrary functions of x, y, z, y', z'. Lie showed that the only case in which every union is turned into a union is the extended point transformation. There may, however, be other transformations which convert some unions into unions. Examples of this sort have been given by Lie in which the unions considered are the curves of a minimal complex or some other Monge equation (II). The general result is as follows:

The most extensive systems of curves that can be converted into curves by element transformations which are not merely extended point transformations are those defined by differential equations of type III, i. e., precisely those characterized by the Meusnier property.

For any system of type III transformations of this sort may be found. If a system not satisfying an equation III is transformed into curves, the element transformation is necessarily an extended point transformation.

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ON THE DISTANCE FROM A POINT TO A SURFACE.†

BY PROFESSOR PAUL SAUREL.

(Read before the American Mathematical Society, April 25, 1908.)

IN a recent number of the BULLETIN,‡ Professor Hedrick has called attention to the fact that the normal distance from a given point to a surface may be a minimum among the distances in every normal plane section without however being a minimum among the distances to the surface. In this connection it may be of interest to observe that for surfaces of a very general type the phenomenon in question can occur only when the given point is a principal center of curvature. In fact, if the equation

* Cf. abstract of the author's paper on this subject in the BULLETIN, vol. 10 (1904), p. 492.

† The theorems established in a previous note (BULLETIN, vol. 13, p. 447) are not new, as I then believed; they are to be found in section 60 of Goursat's Cours d'Analyse. I take this opportunity of acknowledging Professor Hedrick's kindness in calling my attention to the fact.

‡ BULLETIN, April, 1908, p. 321.

of the surface in the neighborhood of the foot of the normal can be written in the form

$$z = ax^2 + by^2 + R, \tag{1}$$

in which the remainder R is of the third or higher order in x and y, and a and b are not both equal to zero, it is easy to establish the following theorems:

I. If the foot of the normal be an elliptic point of the surface, the distance from the nearer (more remote) of the principal centers of curvature may be a minimum (maximum) among the distances in every normal section without being a minimum (maximum) among the distances to the surface.

As illustrations of these two statements may be mentioned the distances from the point $(0, 0, \frac{1}{2})$ to the surfaces

$$z = x^2 + \frac{y^2}{2} + \frac{3}{2}x^2y, \quad \text{and} \quad z = x^2 + 2y^2 - 3x^2y + 3x^4.$$

II. If the foot of the normal be a hyperbolic point of the surface, the distance from a principal center of curvature may be a minimum among the distances in every normal section without being a minimum among the distances to the surface.

The distance from the point $(0, 0, \frac{1}{2})$ to the surface

$$z = x^2 - y^2 + 6x^2y - 3x^4$$

furnishes an illustration of this statement.

III. If the foot of the normal be a parabolic point of the surface, the distance from the principal center of curvature may be a minimum among the distances in every normal section without being a minimum among the distances to the surface.

In this statement the term parabolic point is to be understood as meaning any point for which a or b in equation (1) vanishes.

Professor Hedrick has given an example that falls under this theorem. An equally simple example is furnished by the distance from the point $(0, 0, \frac{1}{4})$ to the surface

$$z = 2x^2 + 10x^2y.$$

IV. The phenomenon discovered by Professor Hedrick cannot occur unless the given point be a principal center of curvature.

To establish the last of these theorems it is sufficient to observe that the square of the distance from a point $(0, 0, d)$ on the

normal to a point (x, y, z) on the surface is given by an expression of the form

$$d^3 + Ax^3 + By^3 + \cdots,$$

in which A and B are constants and the omitted terms are of the third order at least, and that when the given point is not a principal center of curvature the coefficients A and B do not vanish. If, however, the given point be a principal center of curvature, A or B will vanish and an examination of the sign of the non-vanishing coefficient will easily yield the first three theorems.

NEW YORK,
April 7, 1908.

ON THE SOLUTION OF ALGEBRAIC EQUATIONS IN INFINITE SERIES.

BY PROFESSOR P. A. LAMBERT.

(Read before the American Mathematical Society, April 25, 1908.)

I. *Introduction.*

The purpose of this paper is to present a general method for determining all the roots of any algebraic equation by means of infinite series. The method consists in forming three algebraic functions of x from the given equation

$$f(y) = 0, \tag{1}$$

(a) by introducing a factor x into all the terms of (1) except the first and last;

(b) by introducing a factor x into all the terms of (1) except the first and second;

(c) by introducing a factor x into all the terms of (1) except the second and last.

These algebraic functions are expanded into power series in x by Laplace's series. If in these power series x is made unity, the resulting series, if convergent, determine the roots of the given equation. It will be shown that all the roots of the given equation can be expressed in infinite series derived either from the algebraic function formed in accordance with (a), or from the two algebraic functions formed in accordance with (b) and (c). The method presupposes the solution of the two-term equation

$$ay^n + e = 0. \tag{2}$$

In a paper entitled "New applications of Maclaurin's series in the solution of equations" read before the American Philosophical Society, April 3, 1903, the author applied this method to numerical equations. After completing that paper, the author found in an extract of a letter from Cauchy to Coriolis, of January 29, 1837, published in the *Comptes Rendus* of the Paris Academy, an announcement of important results to be obtained by breaking up an equation into two parts and introducing as a factor into one part a parameter which is ultimately to be made unity. In a postscript Cauchy states that he discovered the advantage of making one part a binomial. The author has not been able to find the method sketched in this letter developed.

II. *The Three-Term Equation.*

From the three-term equation

$$ay^n + by^k + e = 0, \tag{3}$$

in accordance with (*a*), (*b*), (*c*) of Article I, are formed the three algebraic functions of x

$$ay^n + by^k x + e = 0, \tag{4}$$

$$ay^n + by^k + ex = 0, \tag{5}$$

$$ay^n x + by^k + e = 0. \tag{6}$$

If equations (4), (5), (6) are written in the form

$$y = \left[-\frac{e}{a} - x\frac{b}{a}y^k \right]^{\frac{1}{n}}, \tag{7}$$

$$y = \left[-\frac{b}{a} - x\frac{e}{a}y^{-k} \right]^{\frac{1}{n-k}}, \tag{8}$$

$$y = \left[-\frac{e}{b} - x\frac{a}{b}y^n \right]^{\frac{1}{k}}, \tag{9}$$

the algebraic functions y can be expanded into power series in x by means of Laplace's series,* which asserts that if

* *Mémoires de l'Académie de Sciences de Paris*, 1777.

$$y = f[z + x\phi(y)], \tag{10}$$

$$\begin{aligned} F(y) = F\{f(z)\} &+ x\phi\{f(z)\}\frac{d}{dz}F\{f(z)\} \\ &+ \frac{x^2}{2!}\frac{d}{dz}\left[\overline{\phi\{f(z)\}}^{2}\frac{d}{dz}F\{f(z)\}\right] + \cdots \\ &+ \frac{x^n}{n!}\frac{d^{n-1}}{dz^{n-1}}\left[\overline{\phi\{f(z)\}}^{n}\frac{d}{dz}F\{f(z)\}\right] + \cdots. \end{aligned} \tag{11}$$

If the algebraic function y of equation (7) is expanded into a power series in x by Laplace's series and x is made unity in this series, and if y_0 denotes any one of the n roots of the two-term equation

$$ay^n + e = 0,$$

there results *

$$\begin{aligned} y = y_0 &+ \frac{b}{ne}y_0^{1+k} + \frac{1+2k-n}{2!}\frac{b^2}{n^2e^2}y_0^{1+2k} \\ &+ \frac{(1+3k-n)(1+3k-2n)}{3!}\frac{b^3}{n^3e^3}y_0^{1+3k} + \cdots \\ &+ \frac{(1+sk-n)(1+sk-2n)\cdots[1+sk-(s-1)n]}{s!}\frac{b^s}{n^se^s}y_0^{1+sk} + \cdots. \end{aligned} \tag{12}$$

If the first n terms of series (12) are placed in a row in regular order, and the following n terms are placed under the terms of this row in regular order, and so on indefinitely, the terms of series (12) will be arranged in n partial infinite series composed respectively of the terms standing in each of the n columns.

This rearrangement of terms is permissible inasmuch as it will appear subsequently that this series is used only when it is absolutely convergent.

The terms of the left-hand partial series are

$$y_0 + y_0^{1+nk}\frac{b^n}{n^ne^n}\frac{(1+nk-n)(1+nk-2n)\cdots[1+nk-(n-1)n]}{n!}$$

* Laplace's series is used only as a matter of convenience. The same expansion can be found by using Lagrange's series, Maclaurin's series, or the multinomial theorem by assuming $y = a_0 + a_1x + a_2x^2 + \cdots + a_ix^i + \cdots$ in equation (7).

$$+y_0^{1+2nk}\frac{b^{2n}}{n^{2n}e^{2n}}\frac{(1+2nk-n)(1+2nk-n)\cdots[1+2nk-(2n-1)n]}{(2n)!}$$

$$(13)\quad +\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .$$

$$+y_0^{1+tnk}\frac{b^{tn}}{n^{tn}e^{tn}}\frac{(1+tnk-n)(1+tnk-2n)\cdots[1+tnk-(tn-1)n]}{tn!}$$

$$+y_0^{1+(t+1)nk}\frac{b^{(t+1)n}}{n^{(t+1)n}e^{(t+1)n}}\frac{[1+(t+1)nk-n][1+(t+1)nk-2n]\cdots[1+(t+1)nk-\{(t+1)n-1\}n]}{[(t+1)n]!}$$

$$+\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .$$

Cauchy's ratio test for convergency is applied to series (13). The ratio of the general term of (13) to the preceding term is

$$(14)\quad y_0^{nk}\frac{b^n}{n^n e^n}\frac{[1+tnk+(k-1)n]\cdots[1+tnk][1-t(n-k)n]\cdots[1-t(n-k)n-(n-k-1)n]}{(tn+1)(tn+2)\cdots(t+1)n}.$$

In each of the two groups of factors in the numerator of the last fraction in this ratio the successive factors differ by $-n$. The first group contains k factors, the second group contains $n-k$ factors.

If the limiting value of this ratio (14) when t becomes infinitely large is numerically less than unity, series (13) is absolutely convergent. This condition gives the inequality between absolute values

$$(15)\quad \frac{e^k}{a^k}\frac{b^n}{n^n e^n}k^k(n-k)^{n-k}<1,$$

whence

$$(16)\quad \left|\frac{b^n}{a^k e^{n-k}}\right|<\frac{n^n}{k^k(n-k)^{n-k}}.$$

In like manner it is proved that the inequality (16) is the condition for the absolute convergence of each one of the remaining $n-1$ partial series into which the infinite series (12) has been broken up.

It follows that when condition (16) is satisfied the infinite series (12) is absolutely convergent.

From equation (7), by means of Laplace's series (11), power series in x can be found for y^n and y^k. If in these power series x is made unity, the resulting series for y^n and y^k are found to be absolutely convergent when condition (16) is satisfied. The values of y^n and y^k make equation (3) an identity. Therefore the value of y in (12) is not only convergent when (16) is satisfied, but it actually represents a root of equation (3). In fact this expression for y determines the n roots of equation (3).

In like manner the series which express the roots of the three-term equation (3) derived from the power series obtained from equations (8) and (9) by Laplace's series are found to be absolutely convergent provided

$$\left|\frac{b^n}{a^k e^{n-k}}\right| > \frac{n^n}{k^k(n-k)^{n-k}}. \tag{17}$$

When condition (17) is satisfied the series obtained from (8) determines $n-k$, the series obtained from (9) determines k roots of equation (3).

Raabe's convergency test* shows that the series which express the roots of equation (3) derived from equations (7), (8), (9) are all convergent for the limiting condition

$$\left|\frac{b^n}{a^k e^{n-k}}\right| = \frac{n^n}{k^k(n-k)^{n-k}}. \tag{18}$$

It follows that all the roots of the three-term equation (3) can always be expressed in infinite series derived either from equation (4) or from the two equations (5) and (6).

Infinite series for determining all the roots of the three-term equation, differing from series (12) only in form, are obtained by Nekrasoff, *Mathematische Annalen*, volume 29, by means of evaluating a definite integral, and by McClintock, in volume 17 of the *American Journal of Mathematics*, by means of his calculus of enlargement. Nekrasoff and McClintock both recognize that their series can be obtained by applying Lagrange's series.

Both investigators found convergency conditions corresponding to (16) and (17) by the methods used in this article. Condition (18) seems to be new.

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* The infinite series $\sum_{n=1}^{n=\infty} u_n$ is convergent if

$$\lim_{n=\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\}>1,$$

Zeitschrift für Mathematik und Physik, vol. 10 (1832).

The results obtained by applying the method of Article I for finding infinite series determining the roots of equations of more than three terms in the following articles, so far as the author knows, are new.

III. *The Four-Term Equation.*

From the four-term equation

$$ay^n + by^k + cy^l + e = 0 \tag{19}$$

in accordance with (*a*), (*b*), (*c*) of Article I are formed the three equations

$$ay^n + by^k x + cy^l x + e = 0, \tag{20}$$

$$ay^n + by^k + cy^l x + ex = 0, \tag{21}$$

$$ay^n x + by^k + cy^l x + e = 0, \tag{22}$$

defining y as algebraic functions of x.

The equations (20), (21), (22) may be written in the form

$$y = \left[-\frac{e}{a} - x\left(\frac{b}{a}y^k + \frac{c}{a}y^l\right)\right]^{1/n}, \tag{23}$$

$$y = \left[-\frac{b}{a} - x\left(\frac{c}{a}y^{l-k} + \frac{e}{a}y^{-k}\right)\right]^{1/(n-k)} \tag{24}$$

$$y = \left[-\frac{e}{b} - x\left(\frac{a}{b}y^k + \frac{c}{b}y^l\right)\right]^{1/k} \tag{25}$$

If the y of equation (23) is expanded into a power series in x by Laplace's series, and after x is made unity the terms of the resulting series are arranged in groups according to the ascending powers of

$$\frac{c}{ne}\left(-\frac{e}{a}\right)^{l/n}, \tag{26}$$

it will be found that the terms of the first group constitute the series which expresses the roots of the three-term equation (3) provided condition (16) is satisfied, and that all the successive groups are convergent when condition (16) is satisfied.

It follows that the series which expresses the roots of equation (19) derived from equation (23) can be written

$$(27)\quad y = X_0 + X_1\frac{c}{ne}\left(-\frac{e}{a}\right)^{1/n} + X_2\frac{c^2}{n^2e^2}\left(-\frac{e}{a}\right)^{2l/n} + X_3\frac{c^3}{n^3e^3}\left(-\frac{e}{a}\right)^{3l/n} + \cdots,$$

where X_0, X_1, X_2, X_3, X_4 $\cdots$ represent convergent infinite series.

The series (27) is convergent provided

$$(28)\quad \left|\frac{c}{ne}\left(-\frac{e}{a}\right)^{1/n}\right| < 1.$$

Condition (28), when both members of the inequality are affected by the exponent n, becomes

$$(29)\quad \left|\frac{c^n}{a^l e^{n-l}}\right| < n^n.$$

The conditions

$$(30)\quad \left|\frac{b^n}{a^k e^{n-k}}\right| \leqq \frac{n^n}{k^k(n-k)^{n-k}}, \quad \left|\frac{c^n}{a^l e^{n-l}}\right| < n^n$$

are therefore sufficient for the absolute convergence of the infinite series expressing the roots of the four-term equation (19) derived from equation (23).

In like manner it is found that the conditions sufficient for the absolute convergence of the infinite series expressing the roots of the four-term equation (19) derived from equations (24) and (25) are

$$(31)\quad \left|\frac{b^n}{a^k e^{n-k}}\right| \geqq \frac{n^n}{k^k(n-k)^{n-k}}, \quad \left|\frac{b^{n-l}}{a^{k-l}c^{n-k}}\right| > \frac{1}{(n-k)^{n-k}},$$

$$(32)\quad \left|\frac{b^n}{a^k e^{n-k}}\right| \geqq \frac{n^n}{k^k(n-k)^{n-k}}, \quad \left|\frac{c^k}{b^l e^{k-l}}\right| < k^k.$$

Either the first condition of (30) or the first condition of (31) and (32) must be satisfied. If it is possible to show that when the first condition of (30) or of (31) and (32) is satisfied the second inequalities can always be satisfied, the possibility of determining the roots of the four-term equation by means of infinite series is established.

The substitution

$$(33)\quad y = z^s,$$

where s is a positive integer, transforms the four-term equation (19) into the four-term equation

$$ay^{ns} + by^{ks} + cy^{ls} + e = 0. \tag{34}$$

The convergency conditions for equation (34) corresponding to conditions (30), (31), (32) are

$$\left|\frac{b^n}{a^k e^{n-k}}\right| \leqq \frac{n^n}{k^k(n-k)^{n-k}}, \quad \left|\frac{c^n}{a^l e^{n-l}}\right| < s^n n^n, \tag{35}$$

$$\left|\frac{b^n}{a^k e^{n-k}}\right| \geqq \frac{n^n}{k^k(n-k)^{n-k}}, \quad \left|\frac{b^{n-l}}{a^{k-l}c^{n-k}}\right| > \frac{1}{s^{n-k}(n-k)^{n-k}}, \tag{36}$$

$$\left|\frac{b^n}{a^k e^{n-k}}\right| \geqq \frac{n^n}{k^k(n-k)^{n-k}}, \quad \left|\frac{c^k}{b^l e^{k-l}}\right| < s^k k^k. \tag{37}$$

The value of s can always be taken so large that the second inequality of (35) is satisfied, or that the second inequalities of (36) and (37) are satisfied simultaneously.

When conditions (35) are satisfied, all the roots of equation (34) are expressed by the infinite series derived from the equation

$$ay^{ns} + by^{ks}x + cy^{ls}x + e = 0; \tag{38}$$

when conditions (36) and (37) are satisfied, all the roots of equation (34) are expressed by the infinite series derived from the equations

$$ay^{ns} + by^{ks} + cy^{ls}x + ex = 0, \tag{39}$$

$$ay^{ns}x + by^{ks} + cy^{ls}x + e = 0. \tag{40}$$

It is therefore always possible to express all the roots of equation (34) in infinite series. The roots of equation (19) are found from the roots of equation (34) by substituting in (33).

IV. *The Five-Term Equation.*

From the five-term equation

$$ay^n + by^k + cy^l + dy^m + e = 0, \tag{41}$$

in accordance with (a), (b), (c) of Article I, are formed the three equations

$$ay^n + by^k x + cy^l x + dy^m x + e = 0, \tag{42}$$

$$ay^n + by^k + cy^l x + dy^m x + ex = 0, \tag{43}$$

$$ay^n x + by^k + cy^l x + dy^m x + e = 0, \tag{44}$$

defining y as algebraic functions of x.

The equations (42), (43), (44) may be written in the form

$$y = \left[-\frac{e}{a} - x\left(\frac{b}{a} y^k + \frac{c}{a} y^l + \frac{d}{a} y^m \right) \right]^{\frac{1}{n}}, \tag{45}$$

$$y = \left[-\frac{b}{a} - x\left(\frac{a}{a} y^{l-k} + \frac{d}{a} y^{m-k} + \frac{e}{a} y^{-k} \right) \right]^{\frac{1}{n-k}}, \tag{46}$$

$$y = \left[-\frac{e}{b} - x\left(\frac{a}{b} y^n + \frac{c}{b} y^l + \frac{d}{b} y^m \right) \right]^{\frac{1}{k}}. \tag{47}$$

If the y of equation (45) is expanded into a power series in x by Laplace's series and after x is made unity in this power series the terms of the resulting series are arranged in groups according to the ascending powers of

$$\frac{d}{ne}\left(-\frac{e}{a} \right)^{\frac{m}{n}}, \tag{48}$$

it will be found that the terms of the first group constitute the series which expresses the roots of the four-term equation (19) provided the conditions (30) are satisfied, and that all the successive groups are convergent when the conditions (30) are satisfied.

It follows that the series which expresses the roots of equation (41) derived from equation (42) can be written

$$y = Y_0 + Y_1 \frac{d}{ne}\left(-\frac{e}{a} \right)^{\frac{m}{n}} + Y_2 \frac{d^2}{n^2 e^2}\left(-\frac{e}{a} \right)^{\frac{2m}{n}} + Y_3 \frac{d^3}{n^3 e^3}\left(-\frac{e}{a} \right)^{\frac{3m}{n}} + \cdots, \tag{49}$$

where $Y_0, Y_1, Y_2, Y_3, \cdots$, represent convergent infinite series.

The series (49) is convergent provided

$$\left| \frac{d}{ne}\left(-\frac{e}{a} \right)^{\frac{m}{n}} \right| < 1. \tag{50}$$

Condition (50), when both members of the inequality are affected by the exponent n, becomes

$$\left|\frac{d^n}{a^m e^{n-m}}\right| < n^n. \tag{51}$$

The conditions

$$\left|\frac{b^n}{a^k e^{n-k}}\right| \leqq \frac{n^n}{k^k(n-k)^{n-k}}, \quad \left|\frac{c^n}{a^l e^{n-l}}\right| < n^n, \quad \left|\frac{d^n}{a^m e^{n-m}}\right| < n^n \tag{52}$$

are therefore sufficient for the absolute convergence of the series expressing the roots of the five-term equation (41) derived from equation (42).

In like manner it is found that the conditions sufficient for the absolute convergence of the infinite series expressing the roots of the five-term equation (41) derived from equations (43) and (44) are

$$\left|\frac{b^n}{a^k e^{n-k}}\right| \geqq \frac{n^n}{k^k(n-k)^{n-k}}, \quad \left|\frac{b^{n-l}}{a^{k-l}c^{n-k}}\right| > \frac{1}{(n-k)^{n-k}}, \tag{53}$$

$$\left|\frac{b^{n-m}}{a^{k-m}d^{n-k}}\right| > \frac{1}{(n-k)^{n-k}},$$

$$\left|\frac{b^n}{a^k e^{n-k}}\right| \geqq \frac{n^n}{k^k(n-k)^{n-k}}, \quad \left|\frac{c^k}{b^l e^{k-l}}\right| < k^k, \quad \frac{d^k}{b^m e^{k-m}} < k^k. \tag{54}$$

Either the first condition of (52), or the first conditions of (53) and (54) must be satisfied.

The substitution

$$y = z^s \tag{55}$$

transforms the five-term equation (41) into the five-term equation

$$az^{ns} + by^{ks} + cy^{ls} + dy^{ms} + e = 0. \tag{56}$$

It is always possible to determine s so that either the convergency conditions for equation (56) corresponding to (52), or those corresponding to (53) and (54) are satisfied. It is therefore always possible to determine the roots of equation (56) by infinite series. The roots of the five-term equation (41) are found from the roots of (56) by substituting in (55).

V. *Conclusion.*

The method used to set up the convergency conditions for the infinite series expressing the roots of the four-term equation derived from the equations formed in accordance with (*a*), (*b*) (*c*) of Article I when the convergency conditions of the three-term equation are known, and to set up the convergency conditions for the five-term equation when the convergency conditions for the four-term equation are known, can be used to set up the convergency conditions for the t-term equation when the convergency conditions for the $(t-1)$-term equation are known.

In fact, the convergency conditions for an equation of any number of terms can be written mechanically.

For the t-term equation

$$f(y) = 0 \tag{57}$$

it is always possible to determine the s of the substitution

$$y = z^s \tag{58}$$

so that the convergency conditions of the infinite series expressing the roots of the t-term equation

$$f(z^s) = 0 \tag{59}$$

derived from the equation formed from (59) in accordance with (*a*) of Article I, or derived from the two equations formed from (59) by (*b*) and (*c*) of Article I are satisfied. The roots of the t-term equation (57) are found from the roots of the equation (59) by substituting in (58).

It is therefore always possible to find all the roots of an algebraic equation by means of infinite series.

LEHIGH UNIVERSITY,
April, 1908.

THE DEDUCTION OF THE ELECTROSTATIC EQUATIONS BY THE CALCULUS OF VARIATIONS.

BY DR. ARTHUR C. LUNN.

(Read before the Chicago Section of the American Mathematical Society, April 17, 1908.)

THE construction of a mathematical theory of classes of physical phenomena for which no detailed mechanical explanation

is available has in several instances led to the enunciation of some general minimal property, usually relating to the distribution of energy and analogous to the principles of least action and of virtual velocities in dynamics. Such a variational "principle" may be regarded either as a concise equivalent for a system of algebraic or differential equations already obtained, or as a form of hypothesis from which those equations may be deduced. The systematic use in the latter sense by Lagrange of the principle of virtual velocities has given his Mécanique Analytique a unity and simplicity far beyond those of most later writings on dynamics.

Corresponding to the equations of electrostatic equilibrium, a number of related forms have been given of a minimizing condition on the energy. These differ among themselves mainly in the character of the variations implied, whether for example variations of a field in which there are charges given or of the charges in a given field.* But in all the forms known to the writer the existence of the electrostatic potential is assumed, while from the point of view of the theory of vector fields it seems more fitting to avoid this assumption if possible, since it is only a special kind of vector field which has a scalar potential. The following deduction according to the formal methods of the calculus of variations, showing how the potential function may appear simply as a lagrangian multiplier, is analogous to the method used by Lagrange for the theory of incompressible fluids, in which the hydrostatic pressure is not introduced at the start as a physical concept, but appears as the multiplier corresponding to the differential condition of invariable volume.†

Adopting the point of view of Faraday and Maxwell, let it be supposed that the electrostatic phenomena can be completely described in terms of a vector point function $\mathbf{E}$, the electric force on unit charge, and a scalar point function ϵ, the dielectric coefficient;‡ that in terms of these the displacement and the volume and surface densities of charge are

$$(1) \qquad \mathbf{D} = \epsilon\mathbf{E}, \quad \rho = \operatorname{div} \mathbf{D}, \quad \sigma = \mathbf{n}'\mathbf{D}' + \mathbf{n}''\mathbf{D}'',$$

where $\mathbf{n}'$, $\mathbf{n}''$ are opposite unit normals of a surface on which the

*See for example: Webster, The Theory of Electricity and Magnetism, § 131; Weber, Differentialgleichungen der mathematischen Physik, I, p. 310.

†Mécanique Analytique, part I, sec. VII.

‡The vector notation and units are those of Heaviside, Electromagnetic Theory.

surface density is the discontinuity in the normal component of $\mathbf{D}$; and that the total energy stored in a given volume by means of the electrostatic strain is

$$W = \tfrac{1}{2}\int \mathbf{E}\mathbf{D}dv. \tag{2}$$

If the total charge on a conductor be supposed to include the surface charge on the boundary between conductor and dielectric, then the charge belonging to any conductor with index i is

$$Q_i = \int \operatorname{div} \mathbf{D}dv_i + \int (\mathbf{n}'\mathbf{D}' + \mathbf{n}''\mathbf{D}'')dS_i,$$

but by transformation of the volume integral to a surface integral this becomes

$$Q_i = \int \mathbf{n}'\mathbf{D}'dS_i, \tag{3}$$

where $\mathbf{n}'$ is the unit normal into the dielectric. If $\mathbf{D}$ should have a discontinuity at a surface within the conductor, the terms in the surface integrals belonging to opposite sides of that surface would be cancelled by corresponding terms from the volume integral, so that the integration need cover only the bounding surface. It is simpler, however, to assume that ϵ and therefore $\mathbf{D}$ are zero everywhere within the conductor, so that the only charge is on the bounding surface, in which case formula (3) still holds.

If ϵ be supposed to vanish within the substance of a conductor and to be a definite physical coefficient different from zero at each point of the dielectric, then the form here to be considered of the condition of equilibrium is that the first variation of W shall vanish for all variations of $\mathbf{E}$ consistent with giving assigned values to the volume and surface densities in the dielectric and to the total charge on each conductor; or, analytically, that with the integration extended throughout the dielectric

$$\delta\int \epsilon\mathbf{E}^2dv = 0 \tag{4}$$

for all variations $\delta\mathbf{E}$ satisfying the conditions

$$div(\epsilon\mathbf{E}) = \rho, \tag{I}$$

where ρ is an assigned function throughout the volume of the dielectric; and

$$\text{(II)} \qquad \epsilon' \mathbf{E}' \mathbf{n}' + \epsilon'' \mathbf{E}'' \mathbf{n}'' = \sigma,$$

where σ is an assigned function over all surfaces of discontinuity in the dielectric; and

$$\text{(III)} \qquad \int \epsilon' \mathbf{E}' \mathbf{n}' dS_i = Q_i,$$

where each Q_i is an assigned constant for the corresponding conductor.

To this conditioned variation of W there corresponds according to Lagrange's method of multipliers a free variation of the modified integral

$$(5) \qquad U = W - \int V \operatorname{div}(\epsilon \mathbf{E}) dv - \sum V_i \int \epsilon' \mathbf{E}' \mathbf{n}' dS_i,$$

where the undetermined function V is the multiplier corresponding to the differential condition (I) and the constants V_i are the multipliers corresponding to the integral conditions (III). But

$$\int V \operatorname{div}(\epsilon \mathbf{E}) dv = \int \operatorname{div}(V \epsilon \mathbf{E}) dv - \int \epsilon \mathbf{E} \nabla V dv$$

and

$$\int \operatorname{div}(V \epsilon \mathbf{E}) dv = - \sum \int V' \epsilon' \mathbf{E}' dS_i - \int (V' \epsilon' \mathbf{E}' \mathbf{n}' + V'' \epsilon'' \mathbf{E}'' \mathbf{n}'') dS_0,$$

where S_0 denotes the surfaces of discontinuity in the dielectric, which in the integration by parts must be taken together with the surfaces of the conductors as the boundary of the realm of integration. Hence U can be written

$$(6) \quad U = \int \epsilon (\tfrac{1}{2} \mathbf{E}^2 + \mathbf{E} \nabla V) dv + \sum \int (V' - V_i) \epsilon' \mathbf{E}' \mathbf{n}' dS_i + \int (V' \epsilon' \mathbf{E}' \mathbf{n}' + V'' \epsilon'' \mathbf{E}'' \mathbf{n}'') dS_0$$

and the condition $\delta U = 0$ for all admissible variations of $\mathbf{E}$ gives the required equations.

By variations $\delta \mathbf{E}$ which vanish at the surfaces S_i and S_0 the first term only of U is affected and gives the condition $\mathbf{E} = - \nabla V$, so that the vector function $\mathbf{E}$ must have a scalar potential V, which according to (I) must satisfy the generalized Poisson equation

$$(7) \qquad \operatorname{div}(\epsilon \nabla V) + \rho = 0.$$

Then with this condition satisfied variations of **E** which vanish at S_0 but not at the surfaces S_i affect only the second term of U and give the condition $V = V_i$, making the potential constant over the surface of each single conductor; and finally variations which at the surfaces S_0 are restricted only by the boundary condition (II) give the condition $V' = V''$, making the potential continuous even at surfaces where its derivatives may be discontinuous.

CHICAGO, ILL.,
May, 1908.

THE FOURTH INTERNATIONAL CONGRESS OF MATHEMATICIANS.

THE fourth international congress of mathematicians was held at Rome, April 6 to 11, 1908, under the efficient management of the Circolo Matematico di Palermo and under the patronage of His Majesty, the King of Italy. Including the ladies who accompanied the members of the congress, the enrollment was more than seven hundred. The list contains the names of the following Americans: Miss E. M. Coddington, H. W. Curjel, E. W. Davis, T. S. Fiske, A. B. Frizell, W. J. Graham, J. G. Hardy, E. A. Harrington, A. S. Hawkesworth, T. F. Holgate, A. Macfarlane, Artemas Martin, C. L. E. Moore, E. H. Moore, Simon Newcomb, G. D. Olds, G. B. Pegram, D. E. Smith, J. M. Van Vleck, W. D. A. Westfall.

The general order of the program provided for sectional meetings in the morning and general conferences in the afternoon. However this order was broken occasionally. The arrangements of the committee on entertainment left nothing to be desired.

The first meeting of the members of the congress was at the reception offered by Professor Tonelli, Rector of the University of Rome, Sunday evening, April 5. This was the beginning of the social part of the occasion. Those who attended the congress will always have pleasant recollections of these receptions and other entertainments. The opening reception was held in the library of the University. The Mayor of the city of Rome, Mr. Nathan, was present. Acquaintances which were to broaden during the week began here. Excellent refreshments were served, and the mathematicians showed that they were not incapable of enjoying this part of the program.

The congress officially opened on Monday, April 6, at a meeting held at the Capitol, in the presence of His Majesty, King Vittorio Emanuele III. The meeting was opened by Mayor Nathan, who welcomed the congress in the name of the city of Rome. The minister of public instruction, Rava, extended a welcome on behalf of the government of Italy. President Blaserna of the committee also made an appropriate address. The session closed with the address of Professor V. Volterra on: "Le matematiche in Italia nella seconda metà del secolo XIX."

The lecturer began by recalling the period of Italian political rejuvenation, in which the whole life of the nation was renewed and the universities reopened. It was then that the instruction in higher mathematics was instituted and entrusted to such mathematicians as Brioschi, Betti, Cremona, Fergola, and Battaglini. Very soon after these followed other mathematicians of no less eminence. The period of original and scientific research in Italy may be said to date from that time. From that moment different Italian schools of mathematics began to be created and to develope.

After drawing a brief comparison between the Italian studies in the first and the second half of the nineteenth century, in order to throw light on the substantial diversity of the surroundings in which they developed, the lecturer passed in review the different Italian mathematical schools, showing their tendencies and their scope.

On account of their research in mathematical physics, which principally absorbed the attention of Betti and Beltrami, he called them the champions of mathematical physics in Italy. He examined the advances in electricity initiated by Betti and continued by a series of other mathematicians, and the researches of Beltrami, who can be placed among those who investigated in a systematic manner the perturbations introduced into the equations of mathematical physics by the hypothesis of a curvature of space.

After having mentioned other investigators in mathematical physics and mechanics, the lecturer passed to another order of investigation which was developed in Italy, viz., that of the theory of functions and allied subjects in analysis. He recalled, in this connection, his address at the Paris Congress on Betti, Brioschi, Casorati, and on the different methods in which these three mathematicians (who were the initiators in Italy of the

study of the theory of functions) conceived the theory itself. He recalled also the various investigators of analysis and dwelt more especially on a branch of research which flourished in Italy a little apart, and which was somewhat forgotten for some years, but which recently acquired general interest and attention, that is research in functions of real variables and their singularities. Dini introduced and diffused their study in Italy and put this theory at the basis of his instruction in the infinitesimal calculus. It then took a double direction, one conducted to research in the field of pure mathematics while the other led to studies which acquired a more philosophic character.

From these researches the lecturer passed to those which, in Italy, are wont to be called geometric. He spoke of the singular struggle, which manifested itself more acutely in Italy than elsewhere, between those who were called analysts and those who were called geometers, but demonstrated the unreality of the distinction which was usually made, and the equivocation on which it was founded. In these days the distinction can only be called a relic of the past.

He examined next the work of Cremona and of his students and disciples, and observed that, forty years after Cremona had begun his teaching, Klein affirmed that Italy had become the real center of geometric research, although this field had been almost ignored in Italy until Cremona initiated his course in higher geometry. The lecturer also examined the researches in hypergeometry, algebraic geometry, and especially the geometric interpretation of the theory of forms. He showed how the most recent Italian geometric investigations, being connected with those of Picard on the theory of algebraic functions, reenter the sphere of the theory of functions. He spoke also of infinitesimal geometry, in which much research has been carried on in Italy, and which makes a noble counterpart to the work in pure and algebraic geometry.

Finally, the history of mathematics developed in the last fifty years was recalled, and the critical publications on the works of Galileo were mentioned.

The lecturer closed by showing that the present time in Italy presents some analogies with a half century ago, and that now, as then, a reform in the program of mathematical studies is necessary. These, in fact, in Italy, are identified with the schools of engineering, for which he urged a more modern organization. He concluded by expressing the hope for a

continuous development in Italian mathematical thought harmoniously united with that of other nations.

In the afternoon the congress met at the Palazzo Corsini and organized. Professor Blaserna was chosen as permanent president, after which the various other officers were named. The first business of the session was the report, presented by Professor Segre, of the committee on the Guccia Medal. There were three competing memoirs, but for various reasons these were adjudged unworthy of the prize. The committee therefore examined the literature published during the time from November 1, 1904, to July 1, 1907, and decided that the prize should go to Professor Francesco Severi for his investigations on the geometry on an algebraic surface. The publications of Professor Severi referred to are the following:

1. "Sulle superficie algebriche che posseggano integrali di Picard della 2ª specie" (*Mathematische Annalen*, volume 61).

2. "Sulla differenza tra i numeri degli integrali di Picard della 1ª e 2ª specie appartenenti ad una superficie algebrica" (*Atti di Torino*, volume 40).

3. "Sulla totalità delle curve algebriche tracciate sopra una superficie algebrica" (*Mathematische Annalen*, volume 62).

4. "Il teorema d'Abel sulle superficie algebriche" (*Annali di Matematica*, series 3, volume 12).

5. "Intorno al teorema d'Abel sulle superficie algebriche ed alle riduzioni a forma normale degl' integrali di Picard" (*Rendiconti di Palermo*, volume 21).

6. "Sul teorema di Riemann-Roch e sulle serie continue di curve appartenenti ad una superficie algebrica" (*Atti di Torino*, volume 40).

7. "Sulle curve algebriche virtuali appartenenti ad una superficie algebrica" (*Rendiconti del R. Istituto Lombardo*, series 2, volume 40).

8. "Osservazioni varie di geometria sopra una superficie algebrica o sopra una varietà" (*Atti del R. Istituto Veneto*, volume 65).

After the reading of the report the Guccia Medal was presented to Professor Severi.

Professor Mittag–Leffler then read his address "Sur la représentation arithmétique des fonctions analytiques générales d'une variable complexe."

He began by recalling that the central point of the theory of analytic functions is, with Weierstrass, the power series

$$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

and that this series is the source from which flows, successively and entirely by transformation and continuation, the analytic function in its entirety. He recalled the theorem of Weierstrass: If an analytic relation, however general or however special, exists between several different power series or their derivatives, this same relation subsists also for the functions in their totality.

The address had for its object to express the principal solutions, obtained during the last ten years, of the following problem: "To form arithmetic expressions of a variable x and of an infinite sequence of constants $c_0, c_1, c_2, \cdots$ which are linear in these constants and have the property of representing the given function $F(x)$, in a domain in which the $c_0, c_1, c_2 \cdots$, once determined, are defined in a unique manner." The first efforts toward the solution of this problem consisted in the investigations of the expressions which represent $F(x)$ not only within the circle of convergence C of $p(x)$ but also on the circumference of C at those points at which $F(x)$ is regular. Borel is the first who arrived at a solution more general, in having obtained an expression valid within a domain B which surrounds, in general, the domain C. Borel's idea that he had obtained, by his summation expression, the power series itself in the case where it diverges, was characterized as a play upon words, all the more intemperate since it gives rise to the illusion, entirely false, that he has been able to extend the limits of the theory of analytic functions beyond those fixed by the classic theory. As one is able, since the expression of Borel is convergent, to perform the same operation on the divergent series as on the convergent series $p(x)$, this scheme implies only the translation, for this special case, of the theorem of Weierstrass just stated.

The complete solution of the problem of the lecture has been obtained finally and in several different ways since the new conception of "the star" (of convergence) was first introduced by Mittag-Leffler in 1898, and one of the solutions, which was obtained by the generalizations of the integrals of Laplace, is attached to the important study of the functions $E_\alpha(x)$ of Mit-

tag-Leffler, as well as to that of the increase of the integral functions in an angle or along different semi-lines.

The lecturer closed by recalling the remarkable property of the function

$$E(x)=\int_s e^z \cdot e^{-e^z} \frac{dz}{z-x},$$

which tends indefinitely and uniformly toward zero when the variable x increases above any limit in the interior of a domain surrounding an angle, however small it may be, and which embraces the infinite part of the real positive axis, but which has the unexpected property of tending uniformly toward zero when the variable increases toward infinity along the real positive axis.

After a short recess Professor Forsyth read his address: "On the present condition of partial differential equations as regards formal integration."

The aim of the lecturer was to call attention to some of what may be called the classic methods of the past, whether recent or more remote, stating that in his opinion their usefulness had not been exhausted. He pointed out the need of a systematic treatment of the subject, since the theory remains fragmentary in form, indeed so much so that each individual success toward the solution of the problem is built up almost independently of its predecessors. The lecture was limited to the consideration of equations of the second order containing one dependent and two independent variables.

Two definitions of general integral were discussed, viz., that of Ampère and that of Darboux. The opinion was expressed that, until the old methods of integration are extended or new and more comprehensive methods given, we are in no position to declare what a comprehensive integral is. The lecturer then proceeded to what he considered the three principal methods of constructing the integral of a partial differential equation of the second order. These methods are the following:

(1) The method of Laplace: It is applicable to strictly linear equations and is effective for them solely when there are two independent variables, but the extension to equations of higher order still remains to be made.

(2) The method of Ampère: The results achieved by Ampère, particularly when regard is paid to the date, were on a

grand scale. Even now his ideas have not received their full development and his discriminating classification remains unamplified to this day. Unfortunately his method, while general in form and spirit, has no compelling force ; success in actual practice depends upon individual skill. His first class of equations, consisting of those whose integrals are expressible in finite terms without partial quadratures, remains as he defined it.

(3) The method of Darboux : When stated in full it can be effectually applied to equations which possess an intermediate integral. The general idea is based on the construction of another equation (still better two other equations) of the same order as the given equation and compatible with it. Moreover the method is progressive, that is to say, when the analysis shows that no compatible equations of the same order exist, similar analysis may lead to the construction of compatible equations of higher order.

One important gap still remains to be filled in this last method. It is true that when the equation is of the required character, the primitive can be constructed by this method, but no method has yet been devised for showing whether a given equation is of the required character. What is needed to fill the gap is one of two things, either some test which will show a priori the lowest order of compatible equations that can be associated with a given equation or else the construction of all the partial equations amenable to the method.

It happens only too often that an equation is not amenable to any one of the methods which have been mentioned. In that case the only plan of deriving information concerning the primitive is to have recourse to Cauchy's existence theorem. Usually this presents itself in inconvenient form, but at present we can do no better.

Some suggestive results have been obtained by proceeding from a different initial standpoint, using once more such classic processes as those of Euler and Lagrange. The principle of such methods is to regard the primitive as the origin and the partial equation as the consequence of the relation between the primitive and the equation.

We have thus seen that in this subject we can take only a few steps before reaching the boundary of present knowledge.

On Tuesday morning the four sections met for the first time, organized, and proceeded to the reading of special papers.

Each morning for the remainder of the week was devoted to the sectional meetings. On Tuesday afternoon two general papers were read. The first was by Professor Darboux on "Les méthodes et les problèmes de la géométrie infinitésimale."

The lecturer first gave a short account of the origin of infinitesimal geometry. Like other branches of human knowledge it had its origin in a practical problem, viz., that of mapping the surface of the earth. Lambert was the first to propose the problem, in full generality, of mapping the surface of the earth on a plane in such a way that infinitesimal elements shall be similar. It gave rise to some beautiful researches of Lambert, Euler, and Lagrange, but the complete solution was first given by Gauss in 1822. This was followed in 1827 by his Disquisitiones. In this last are found many ideas destined to great development in the modern theory of surfaces.

Professor Darboux next spoke of the activity in France during the same period and made special mention of the works of Monge, Dupin, Chasles, and Lamé, and the influence which they had on the development of infinitesimal geometry. Mention was also made of the geometers who followed these, and Ossian Bonnet was characterized as one to whom his works give almost the role of creator.

Having thus briefly sketched the history of the subject, the lecturer proceeded to what he considered the proper method of infinitesimal geometry. He said the method should be that of analysis which makes use of coordinate axes. The research should always be vivified and inspired by the geometric spirit. The method should not be followed blindly, for this being a grand highway will lead the more securely to the end desired, but the other roads of travel indeed have their charm. He advised the complete and frank introduction of imaginaries.

He then passed to the discussion of some of the unsolved problems in infinitesimal geometry.

1. The curves of constant torsion are of great importance in the study of surfaces applicable to a paraboloid of revolution, as these surfaces can be derived by a geometric construction from imaginary curves of constant torsion. Now it would be a problem worth the effort to determine all those curves of constant torsion which are algebraic or even unicursal. Attempts have been made at the problem, but a general solution is yet to come.

2. Bour announced that by applying the celebrated method of

Lagrange, the variation of constants, he had obtained all the surfaces applicable to any surface of revolution whatever, and in particular to a sphere. The manuscript was lost in a fire. Many geometers have sought to find the solution of Bour, but as yet no one has succeeded.

3. The initial conditions selected in the discussion of the problem of Cauchy should be as general as possible. For example, the following are suggested: If it is a question of an equation with two independent variables, the surface should be restricted to pass through a given curve when the equation is of the first order, but if the equation is of the second order the surface should be tangent the whole length of a curve to a developable surface. This method should lead in a precise manner to the notion of characteristics.

4. The above problem has been solved completely for minimal surfaces, but it is not to be confounded with the very different problem which has for its object the determination of a continuous minimal surface which passes through a given closed curve. Much has been done, but much still remains to be done with this latter problem.

5. Along with the problems of mapping, the problem of Tchebychef, somewhat generalized, was considered. Imagine a net, of any form and dimensions, composed of two series of threads attached firmly at their points of intersection so that the angles but not the sides of the meshes may vary. Suppose a solid bounded by any surface whatever be dropped into the net. Determine the form which the net assumes. In case the solid is spherical the solution can be connected with the determination of surfaces of constant curvature.

6. Some of the first and most elementary problems connected with the quadratic differential forms are yet to be solved. Here the great discoveries of Lie on the theory of groups have found and will continue to find beautiful applications.

7. The problem of the deformation of surfaces has been solved for the paraboloid, quadrics of revolution, and quadrics tangent once to the circle at infinity. Can the results be extended to quadrics in general? The answer, whether positive or negative, will be interesting.

The lecture closed with showing the important part which infinitesimal geometry plays in the study of partial differential equations.

Following Professor Darboux, Professor von Dyck presented a report: "Über die mathematische Encyklopädie."

On Wednesday afternoon Professor Newcomb read his address: "La théorie du movement de la lune: son histoire et son état actuel."

Since the law of gravitation was enunciated it has been the work of many mathematicians to compute theoretically the movement of planets, and in nearly all cases the theoretic movement agrees with the observed movement, but there are two noted exceptions, viz., Mercury and the moon. In the case of Mercury the difference can be accounted for by the presence of an unknown body between Mercury and the sun, but in the case of the moon we are confronted by an enigma. The importance of this divergence can be recognized after we have sketched the present state of the theory of the movement of the moon.

The law of gravitation is expressed by means of three differential equations, with time as the independent variable. The deductions are obtained by integrating these equations. The integrals give the coordinates of the body in terms of t and six arbitrary constants. The general solution in the case of the moon is not possible, but on account of the smallness of certain elements we can obtain a solution in the form of an infinite series arranged in powers and products of these elements. The coefficients are periodic functions of the time and we have solved the problem when we have calculated these. The method of Delaunay for handling these equations was discussed. He reduces the solution to the execution of a series of algebraic operations, comprising substitutions repeated without end but always approaching more and more nearly the exact value of the variables. This series for the coefficients, however, is so slowly convergent that it is not sufficient for the needs of modern astronomy.

There is a method used by Hansen which consists in substituting the numerical values for the constants, but since the derivatives cannot be calculated we must consider this method as unsatisfactory.

However there is a method which, in the speaker's opinion, has all the exactness demanded by the astronomy of our time. This is the method initiated by Euler and completed by G. W. Hill nearly a century later. Considering the mean movement of the moon about the earth and of the earth about the

sun as known, the coordinates can be expressed in a series arranged according to powers of e, e' (the eccentricities of the two orbits), and $\gamma = \sin \frac{1}{2} I$. Then, y being such a coordinate,

$$y = P_0 + eP_1 + e'P_2 + \gamma P_3 + e\gamma P_4 + \cdots,$$

where the P's have the form

$$P = \sum \begin{matrix} \sin \\ \cos \end{matrix} (A + Bt).$$

Euler sought to develop separately the values of P, but found difficulty in introducing e, e', γ in B. Hill succeeded in developing P_0 in terms of the mean movement. He also gave a method for treating that part of the motion in perigee, independent of e, e', γ. E. W. Brown has attacked the problem also and has conquered its difficulties. The degree of exactness of Brown's work is all that could be desired.

These investigations all assumed that the sun is the only source of perturbation, but the planets also exert a small force which causes an acceleration of the mean movement of the moon, and there is besides a disagreement between theory and observation of which William Ferrel has found the partial cause in the effect of the moon upon the tides. But even this does not account for the entire disagreement. To explain the variations in the mean movement of the moon by the action of the tides, it would be necessary to suppose variations of almost a minute in our measure of time during the last two centuries. But the transits of Mercury seem to show that the variations cannot be more than a few seconds.

The lecturer closed by saying that the enigma seemed to him one of the most important and most interesting problems in celestial mechanics.

After a short recess Professor Lorentz gave his address: "Le partage de l'energie entre la matière ponderable et l'éther."

Kirchhoff reached the conclusion that the energy of radiation which exists in a unit volume of ether, so far as it depends upon a wave length comprised between the limits λ and $\lambda + d\lambda$, can be represented by an expression of the form

$$F(\lambda, \tau)d\lambda,$$

where F is independent of the special properties of the body. The lecture was especially concerned with the form of the function F. Wien has given to the function the form

$$F(\lambda, \tau) = \frac{1}{\lambda^5} f(\lambda\tau),$$

where instead of two independent variables λ, τ we have only the product $\lambda\tau$, but the form of f is still undetermined.

Professor Lorentz pointed out that Maxwell and Boltzmann had made use of two important branches of mathematics, viz., the calculus of probability and geometry of n dimensions. After showing the importance of these, he proceeded to limit the problem to considering a small mass M, composed of innumerable atoms animated and in motion in a rectangular box; further there are some charged particles or electrons free or held captive in the interior of the atoms. These electrons take part in the calorific movement of the atoms and should be regarded as the source of radiation.

The method used to treat the problem was that due to Gibbs. The state of the ether when some movable electrons are present is expressed by a system of partial differential equations seemingly very different from those of Hamilton, but which can be reduced to those by first establishing a theorem analogous to that of least action.

It is found that one meets a difficulty in applying the Gibbs method because the number of coordinates which define the electric field in the ether is infinite. It is necessary, therefore, to replace this system by a virtual system which has n degrees of freedom and treat the real system as the limit of this as n becomes infinite.

Finally the form of the function of radiation is found to be

$$F(\lambda, \tau) = \frac{16\pi\alpha\tau}{\lambda^4}.$$

This, however, is only valid for great wave lengths, but is definite enough for these. α is a universal constant which can be computed by making measurements on the infra-red rays. Planck by an entirely different method has obtained the formula

$$F(\lambda, \tau) = \frac{8\pi ch}{\lambda^5} \cdot \frac{1}{e^{ch/2\alpha\lambda\tau} - 1},$$

which for long wave lengths agrees with the previous result.

The fact that F is independent of the special properties of the body is accounted for by the energy of agitation of the constituents, represented by $a\tau$, which determines the intensity of radiation in the ether.

The lecturer closed by considering the merits and defects of the various methods for determining F.

On Wednesday evening a reception was given in the museum of sculptors at the Capitol, by the municipality of Rome. The halls were brilliantly lighted and beautifully decorated and their capacity, though very large, was tested. Mayor Nathan and other officials of the city were present. This reception was no less a delight to the members than the one on Sunday evening and the halls were not cleared until after midnight.

The usual program was followed Thursday morning, but in the afternoon the general addresses were suspended, and the congress adjourned to the Palatine, where a most delightful season was spent in viewing the many things of interest contained there. Guides who spoke different languages were furnished. After spending some time in inspecting the old Roman remains we were conducted to a beautiful spot on the Hill where a fine lunch was served. Everyone went away with the feeling that these non-sessions were among the most delightful parts of the congress.

Thursday evening all were provided with tickets to the concert at the Amphitheatre Corea. Those who loved music certainly enjoyed a treat and the others did not regret going.

Friday afternoon, Professor Poincaré being ill, his address was read by Professor Darboux. The title was: "L'avenir des mathématiques."

The author first examined the tendencies of the past and the present in order to be able to predict the future from these. He finds that there are two forces which influence mathematical thought. The first is the demand which physicists and engineers make on the mathematician for the solution of their problems. The second is the demand for the economy of thought. This economy is effected by organization and in many other ways; even the elegance to which mathematicians attach so much importance is attributed to this economy of thought.

In this economy it is not sufficient to give models to be followed. It is necessary that once knowing a given piece of reasoning, it can be repeated, in substance, in a few words. That is to say, a special choice of words is necessary. This choice of words has also another merit, it enables one to see that demonstrations made concerning one class of objects are

valid for another also. Two good examples of this are the words group and invariant.

Having determined that these are the forces which influence mathematical thought to-day, the author proceeded to make predictions concerning various branches of mathematics.

Arithmetic: The development of arithmetic is now much behind that of algebra and analysis, mainly because of the discontinuity of its elements. The future of arithmetic therefore lies in the direction of making use of the advances in algebra and analysis in order to develop itself. For example, the parallelism between algebraic equations and congruences is almost perfect. Certainly this can be carried to completion.

The theory of primes in arithmetic seems to lack unity almost entirely, but without doubt unity will be produced by the consideration of a family of transcendental functions which permit, by the study of their singular points and the application of the methods of Darboux, the calculation asymptotically of certain functions in very great number.

Algebra: In the study of algebraic equations there still remains the problem of a system of invariants which do not change sign when the number of roots remains the same.

If one forms power series representing functions which admit the roots of an algebraic equation for singular points, the coefficients of the terms of higher order will furnish one of the roots with an approximation more or less close. Here is the germ of a process for numerical calculation of which a systematic study could be made.

The theorem of Gordan in the theory of invariants is yet to be extended.

Differential equations: In studying the functions defined by a differential equation we will not be satisfied until we have found groups of transformations which play the same rôle with respect to the differential equation that the groups of birational transformations play with respect to algebraic curves.

The lecturer emphasized the qualitative discussion of the curves defined by a differential equation. This is analogous to the investigation of the number of roots of an algebraic equation.

Partial differential equations: Here the idea of Fredholm can be applied to all the equations of the linear form. This remains to be carried to its completion.

Theory of functions: Here the theory of functions of more than one variable will occupy the attention; a mere analogy to

functions of one variable is not sufficient. We must search for the things which will throw light on the difference between the theory for one variable and for several. For example, what is to take the place of conformal representation?

Theory of groups: The theory of continuous groups being much more advanced than the theory of Galois groups, it remains for the latter to make use of the analogies with the former.

Geometry: Analysis situs has been of great aid to mathematicians and still awaits to be completely constructed for space of more than three dimensions. When this is done we shall have an instrument which shall permit us really to see in hyperspace. The study of groups in geometry has contributed much to its growth. The study of groups of curves on a surface, analogous to the groups of points on a curve studied by Brill and Noether, has never been worked out.

Postulates: It seems at first that this field is well limited and that there is nothing to do when the inventory is completed. But when all are enumerated, there will be different ways of classifying them and each new classification will be of interest to the philosopher.

After a short recess Professor Picard gave his address: "L'analyse dans ses rapports avec la physique mathématique."

The principal object of the lecture was to show the mutual relation between physics and mathematics, that is to show that each derived much benefit from the other.

In the seventeenth century the development of kinetics and dynamics gave birth to the greatest advances in analysis; from that time dates the real beginning of modern analysis, it really came from mechanics. The origin of the notion of derivative is involved in the idea of the movability of things and of the rapidity with which phenomena occur. It was a decisive epoch in the history of mathematics when the development of mechanics conducted to the postulation that infinitesimal changes depend uniquely on the actual state of the system.

In the eighteenth century the development of mathematics in its most essential points was identified with that of mechanics. Clairaut in his study of the form of the earth considered, for the first time, curvilinear coordinates, d'Alembert used the two fundamental equations of the theory of functions and saw the significance of $\sqrt{-1}$. We find also functions of a complex variable considered by Euler and Lagrange.

The theory of potential and the analytic theory of heat have contributed much to mathematics. In these we often find also suggestions of the method of solution of the mathematical problem. Thus in this case physics has rendered a double service to mathematics.

There are also many examples of the service which mathematics has rendered to physics. The monumental work of Green on the applications of analysis to electricity and magnetism is an excellent example. In the study of the theory of waves we find also that differential equations have rendered much service; for example, in the theory of heat they showed that any variation is felt instantly in all directions and that therefore we cannot speak of the velocity of propagation. The influence which formal mathematics has had on physics and mechanics are instanced also by the equations of Laplace and the principle of least action.

The knowledge of the integrals of partial differential equations has caused some noted paradoxes in physics to vanish. The modern theory of functions seems at first to present but little of interest to the physicist, but extended study shows the theory of analytic functions to be of extreme importance to the mathematical physicist. The questions of domain of existence and of the prolongation of integrals have no less interest for the physicist than for the analyst. Indeed it seems to us that analysis is an indispensable instrument to the physicist and in some cases even a precious guide.

On Friday evening Professor Störmer gave his address: "Sur les trajectoires des corpuscles électrisés dans le champ d'un aimant élémentaire, avec application aux aurores boréales." The lecture was illustrated by a set of lantern slides. It was a résumé of a large memoir on the same subject, published in 1907 in the *Archives* of Geneva.

Partly by analysis and partly by the powerful methods of numerical and graphic integration of differential equations, Professor Störmer has succeeded in finding the properties of the trajectories in question. The numerical calculation represents a labor of nearly 5,000 hours. The author showed afterwards how the results found sufficed to explain a whole series of details in the remarkable experiments of M. Birkeland, in which he exposed a magnetic globe to a pencil of cathode rays, as explaining the principal character of the aurora borealis, for example

the zones of auroras, the appearance in the night, the auroral rays and the remarkable phenomena of draperies of auroras.

The results of the applied analysis now make very probable the hypothesis of M. Birkeland, according to which the auroras are due to cathode rays, or something analogous, emitted by sun spots and meeting the atmosphere of the earth under the action of terrestrial magnetism.

On Saturday afternoon, Professor Veronese being ill, his lecture on "Geometria non-archimedea" was not read, but it will appear in full in the Atti del Congresso. The congress proceeded to the consideration of recommendations made by the various sections. The following resolutions were adopted:

The congress, having recognized the importance of an accurate examination of the program and the methods of instruction in mathematics in the secondary schools of the various nations, appoints Professors Klein, Greenhill, and Fehr as an international committee to study the question and to report upon it at the next congress.

"Section III (mechanics), after an exchange of views in which the importance of a unification of the notation of vectors was recognized, proposes to the congress the nomination of an international committee for the study of this question. The president of this section for the session of April 11 proposes to the congress to adopt its committee of organization to constitute this commission, and submits the list of names."

"The congress votes that at the next congress the constitution of the International Association of Mathematicians be presented."

"It is clear, from the exchange of views in Section III-b, that it would be highly desirable to effect a closer union between those who are occupied in perfecting mathematical methods and those who make practical applications. To this end the section urges that mathematics applied to the science of engineering be the object of a special session at the next congress. Section III-b further proposes the appointment of an international commission, which shall have charge of the work of this new section. The composition of this international commission shall be fixed by the bureau of the fourth congress."

"The Fourth International Congress of Mathematicians in Rome considers as a matter of maximum importance to the mathematical sciences, both pure and applied, the publication of

the complete works of Euler. The congress greets with approval the initiative taken by the society of Swiss naturalists, and votes that the great work be taken up by the society itself with the help of the mathematicians of other nations. The congress urges the international association of the academies, and especially the Academies of Berlin and St. Petersburg, of which Euler was a celebrated member, to aid the enterprise."

The congress unanimously accepted the invitation of the Cambridge Philosophical Society to hold its fifth meeting at Cambridge in 1912.

Professor Mittag-Leffler, on the part of the Swedish mathematicians and the King of Sweden, invited the Congress to hold its sixth meeting at Stockholm in 1916. This of course could not be voted upon at this meeting.

Professor Darboux in the name of all the members of the congress thanked the committee and all those who had had a part in making the fourth international congress so important and so pleasant. The president then closed the congress officially.

While the official end of the congress was Saturday, there was still one more very pleasant non-session to be held on Sunday. All were furnished with tickets to Hadrian's Villa and Tivoli. The first stop was at Hadrian's Villa. Carriages were ready to take those who did not care to walk from the station to the villa. On entering the ruins, we found refreshments, provided by the municipality of Tivoli, ready and waiting to be served. After spending about two hours here, we proceeded to Tivoli, where a banquet awaited us. The banquet closed with toasts in Italian, French, German and Latin.

The afternoon was spent in visiting the cascades and the Villa d'Este. The returning trains arrived in Rome about 8 P. M. This was the unofficial but real close of the congress.*

C. L. E. MOORE.

ROME, *May*, 1908.

* A report of the sectional meetings of the Congress will appear in the October BULLETIN.

SHORTER NOTICES.

Encyklopädie der Elementar-Mathematik. Ein Handbuch für Lehrer und Studierende. Von HEINRICH WEBER und JOSEF WELLSTEIN. Zweiter Band, *Elementare Geometrie.* Bearbeitet von HEINRICH WEBER, JOSEF WELLSTEIN, und WALTHER JACOBSTHAL. Leipzig, Teubner, 1905. 8vo. 280 figures. viii + 602 pp. Dritter Band, *Angewandte Elementar-Mathematik.* Bearbeitet von HEINRICH WEBER, JOSEF WELLSTEIN, und RUDOLF H. WEBER. Leipzig, Teubner, 1907. 8vo. 358 figures. ix + 666 pp.

EVERY live teacher of secondary school mathematics is aware of the superficial character of most text-books. In the nature of the case, teaching for younger pupils must exclude far more than it presents. But if the teacher himself restricts his study to the range prescribed for pupils, very little mathematical interest is kindled in his classes. Practically the same result is reached if the sole scientific interest of the teacher is in fields remote from his pupils' studies. The authors of this three-volume encyclopedia of elementary mathematics plan to intensify by fundamental criticism, and revivify by extensive applications to questions of physics, the interest of the young teacher in his every-day work. This does not conflict by any means with the programme of modern universities — to train the future teacher by research in some region on the frontiers of scientific knowledge. Rather it supplements that programme, and strengthens the position of its champions, by showing how to apply the method of the university seminar to the problems of the school-room.

The book on geometry begins with a critical and historical survey of the notions point, straight line, surface, plane, parallel. The antithesis is developed between what the authors call natural and logical geometry, the latter reached only by a limit process of idealization. Most appropriate is then the quite full examination of a second system of geometrical objects, the totality of spherical surfaces that contain a fixed point O; for in this system there is an exact correspondence to the objects of ordinary euclidean geometry, while the images are radically different. Straight lines are replaced by circles through O, and planes by spheres through O. Tried upon this material, the Hilbert axioms of connection and of order lose their appearance of artificiality and become novel observations

of fact. Then an exchange (duality) of sphere for point, sheaf of spheres for line, etc., serves to shake loose the habitual prepossession that point must mean always one thing, and that the truth of the theorems of Euclid's geometry is mystically dependent upon something more than the axioms. Soon is introduced also the three-fold infinity of circles in a plane: each circle is called a point, each pencil of circles a line, and each "Bündel" a plane. It is easy thence to foresee how there must arise elliptic and hyperbolic non-euclidean geometries in a plane or on a spherical surface.

By easy stages the reader is led to Hilbert's inquiries respecting the independence of axioms, and to the "pathological" systems constructed for the purpose of establishing that independence. The climax of this fascinating chapter is reached in the paragraph 14, where metageometry is considered in its relation to philosophy, especially to Kant. It will interest the casual reader to note (page 144) the explicit opinion: "Necessities of intuition there are none; necessity can lie only in acts of the intellect;" while a footnote states that on this point there is a difference of opinion between the two editors.

Projective geometry is treated no less carefully as to its axioms, conics are considered in their main features — and shown to be central projections of circles, and projective metric concludes this third section, to which is added a well chosen list of references.

Planimetry (pages 220–301), shows the same influence of Hilbert's "Foundations." In particular it contains a section that will be welcome to very many teachers of geometry, upon the number π and the history of circle measurement. Inversion with respect to a circle, and the problem of Apollonius appear as extensions of the traditional school-book material.

Trigonometry, plane and spherical, is given a concise discussion. New to many will be the division of formulas into those of the first order and those of the second, and the generalization of the spherical triangle (Gauss and Study), with Study's theorem that the totalities of proper and of improper spherical triangles form two discrete continua, with no continuous transition from the one to the other by moving the vertices freely in the surface of the sphere. Three points of the surface denote a great number of different triangles when sides and angles are unrestricted in magnitude, and for both the distinction is maintained between positive and negative. Adopting any modulus,

as 2π, 4π, 6π, etc., renders the number of non-equivalent classes of triangles with three fixed vertices finite, and Study's theorem is valid for all moduli.

Brief but artistic sections on analytic geometry of the plane (76 pages) and of space (72 pages) include much of value, as the elements of integration for volume, and the rotation groups of regular solids. A good index, and a full supply of clear diagrams, make this a valuable book of reference even for teachers who will read it but infrequently.

Of the third volume it is not too much to say that it contains a most valuable presentation of physical theories for mathematical teaching. That it is kept free from overloading of theory is seen perhaps in the fact that continuity and discontinuity are not mentioned in the index, nor critical states of matter. Half the volume is physics, vector geometry, analytical statics, dynamics, electricity and magnetism, and electromagnetism. Of the remainder, maxima and minima in geometry and capillarity fill 43 pages; probability and least squares, 40 pages; and a full and suggestive book on graphical statics the concluding 240 pages.

In a note appended to this volume, H. Weber reverts to the Mengenlehre of the first volume, cites Russell's paradox on the class of classes that do not contain themselves (which he identifies with one of Kant's antinomies); and gives an outline discussion of finite aggregates, free from objections, as he believes. These volumes certainly constitute a valuable work for every reference library. H. S. WHITE.

Leçons sur l'Intégration et la Recherche des Fonctions Primitives. Par HENRI LEBESGUE. Paris, Gauthier-Villars, 1904. 8vo. viii + 138 pp.

SINCE the publication of Lebesgue's thesis in 1902 the originality and power of his methods have attracted increasing attention to the field in which he and Baire have made such important contributions. They have given to the study of discontinuous functions an impulse which is apparent on the most cursory survey of current mathematical periodicals, and of such recent treatises as those of Young and Hobson.

The present volume, one of the series of monographs published under the direction of Borel, reproduces a course of twenty lectures delivered at the Collège de France on the Peccot foundation. Within such limits one could hardly ex-

pect a complete treatment of so large a subject as the title suggests, and in fact the author restricts himself to real functions of a single real variable. The only applications considered are connected with the problem of primitive functions and the rectification of curves, the author's researches on trigonometric series being left for a subsequent volume of the same collection. Even with this narrowing of the field the material seems almost too abundant and the reader arrives at the end of each chapter almost out of breath from the rapid pace which the author sets. One is inclined to wonder whether this haste of treatment did not extend to the editing of the volume. There is an unusual number of misprints and inaccuracies in formulas, particularly in the last chapter, and in three or four instances rather extended proofs are faulty. Lebesgue has himself recently indicated some important corrections in the *Atti della Reale Accademia dei Lincei* (series 5, classe di scienze fisiche, etc., volume 15, 1906). It should be added however, that the theorems concerned are true as stated; the inaccuracies are in the proofs. In some places brevity interferes with clearness, particularly in statements where conditions must be supplied by the reader. But, apart from these minor defects, one cannot but feel that this is a brilliant piece of work, full of originality, ingenious in its methods, important in its results.

To a certain extent the historical development of the subject is followed, beginning with definitions of integration deduced from the work of Cauchy and Dirichlet and terminating with an exposition of the method developed by the author. Everywhere the theory of point sets is of fundamental importance, and in particular the notion of the measure of a set of points. By a happy extension and simplification of Borel's definition Lebesgue gives to this term a meaning which is especially adapted to functional operations involving an enumerable infinity of elements. This may be briefly indicated as follows for a linear set E belonging to a segment AB: The points of E are enclosed in a set of intervals whose number is finite or enumerably infinite;* if we consider the sum, or the limit of the sum, of the lengths of these intervals it is obvious that for all possible sets enclosing E this sum has a lower limit $m_e(E)$; this is called the exterior measure of E. The interior measure

* If we omit the words "or enumerably infinite" this becomes the definition of measure in Jordan's sense. Lebesgue's measure includes both Jordan's and Borel's.

$m_i(E)$ is defined as the difference between the length AB and the exterior measure of the set of all points of AB which do not belong to E. If the exterior and interior measures are equal the set is said to be measurable, its measure being the common value of $m_e(E)$ and $m_i(E)$. The extension to sets in more than one dimension is obvious. The generality of this definition may be inferred from the fact that no non-measurable sets have yet been constructed by processes which do not involve such "idealistic" notions as are involved in an arbitrary choice among an infinite number of elements, though existence proofs of the latter kind have been given recently by Lebesgue, Vitali, and Van Vleck. The first-named has, however, constructed sets that are not measurable in Borel's sense.

A first application of the idea thus introduced is to be found in the formulation of the condition for the existence of an integral in Riemann's sense which requires, in the case of a limited* function, that its points of discontinuity in the integration interval form a set of measure zero. But the chief importance of this notion for Lebesgue's work arises from the part it plays in his definition of the integral, a definition which applies to all limited measurable functions, *i. e.*, functions such that for any two numbers α, β the points on the x-axis in the interval $[a, b]$ for which $\alpha < f(x) < \beta$ form a measurable set. The generality of this category of functions is apparent from what is said above as to the existence of non-measurable sets. With minor changes of notation Lebesgue's definition may be thus given: Divide the interval of variation of $f(x)$ into n subintervals by means of the numbers

$$y_0 < y_1 < y_2 < \cdots < y_n = Y,$$

and designate by $m(E_i)$ the measure of the set of points on the segment $[a, b]$ of the x-axis for which $y_{i-1} < f(x) < y_i$, and by $m(E_i')$ the measure of the set for which $f(x) = y_i$. Then the common limit as n becomes infinite of the two sums

$$S = \sum_{i=1}^{i=n} y_i m(E_i) + \sum_{i=0}^{i=n} y_i m(E_i'), \quad s = \sum_{i=1}^{i=n} y_{i-1} m(E_i) + \sum_{i=0}^{i=n} y_i m(E_i')$$

is the integral in Lebesgue's sense of $f(x)$ from a to b.

* Lebesgue uses the term limited for functions all of whose values in the interval considered lie between two constants A and B; a function is finite if it has a finite value at every point of the interval.

If $f(x)$ is integrable in Riemann's sense then this new definition of the integral coincides with the ordinary one. Perhaps the relation of these two ideas is most clearly shown by means of the geometric conception of the integral as the difference between the two-dimensional measure of the set of points in $[a, b]$ having positive ordinates for which $0 < y < f(x)$, and that of the set of points for which these signs of inequality must be reversed. If these measures exist according to Jordan's definition, their difference is the integral in Riemann's sense; the insertion of the words "or enumerably infinite" in the right place in the definition of measure gives us the Lebesgue integral.

In so brief a notice as this it is impossible even to indicate the many applications which the author makes of this new and powerful instrument of analysis. There still remain unsolved cases of the problem of the primitive function, *i. e.*, the problem of determining a function whose derivative is given, but the solution is found in many cases where the ordinary integral cannot be used. In particular the question is settled whenever the given function is limited, or when it is known that the primitive function must be of limited variation. And in the case of rectifiable curves the Lebesgue integral gives their length whenever the functions $x(t)$, $y(t)$, $z(t)$ which define the curve have limited derivative numbers.

The treatment of several topics connected with the main subject should be mentioned, especially the chapters and sections on functions of limited variation and on derivative numbers. The volume closes with an admirably clear and concise note on point sets and transfinite numbers. D. R. CURTISS.

NOTES.

AT the meeting of the London mathematical society held on May 14 the following papers were read: By P. A. MACMAHON, "On the invariants of the general linear homogeneous transformation in two variables"; by H. HILTON, "On the order of the group of isomorphisms of an abelian group."

A NEW academy of sciences has been established in Finnland, with seat at Helsingfors. Two sections have been organized, one consisting of mathematics and the physical sciences, the other consisting of philology and philosophy.

THE French association for the advancement of science will hold its annual meeting at Clermont Ferrand during the first week in August.

THE following advanced courses in mathematics are offered during the academic year 1908–1909:

JOHNS HOPKINS UNIVERSITY. — By Professor F. MORLEY: Higher geometry, two hours; Dynamics, two hours (first half year); Theory of functions, two hours (second half year); Reading course, one hour; Seminary, one hour. — By Dr. A. COHEN; Introduction to the theory of functions, two hours; Differential equations and differential geometry, two hours. — By Dr. A. B. COBLE: Theory of groups, two hours.

UNIVERSITY OF PENNSYLVANIA. — By Professor E. S. CRAWLEY: Modern analytic geometry, two hours; Theory of numbers, three hours. — By Professor G. E. FISHER: Differential equations, two hours; Elliptic functions, three hours (first half year); Topics in the theory of differential equations, three hours (second half year). — By Professor I. J. SCHWATT: Infinite series and products, two hours. — By Professor G. H. HALLETT: Lie's theory of continuous groups, three hours (first half year); Galois's theory of equations, three hours (second half year); Groups of finite order, three hours. — By Professor F. H. SAFFORD; Curvilinear coordinates, three hours; Partial differential equations, three hours. — By Dr. O. E. GLENN; Invariants, three hours.

UNIVERSITY OF BOLOGNA. — By Professor C. ARZELA: Dirichlet's principle, calculus of variations, three hours. — By Professor L. DONATI: Mathematical theory of elasticity, physical optics, three hours. — By Professor S. PINCHERLE: Algebraic functions and their integrals, elliptic functions, analytic functions represented by definite integrals; three hours.

UNIVERSITY OF CATANIA. — By Professor M. DE FRANCHIS: Geometry on algebraic curves, Riemann's surfaces and abelian integrals, inversion, four and a half hours. — By Professor S. LAURICELLA: Optics, four and a half hours. — By Professor G. PENNACCHIETTI: Elliptic functions with applications to mechanics, four and a half hours. — By Professor C. SEVERINI: Application of Lie's theory of continuous groups to differential equations, researches of Picard and Vessiot, four and a half hours.

UNIVERSITY OF GENOA. — By Professor G. FUBINI: Theory of continuous groups and of automorphic functions, three hours. — By Professor G. LORIA: Infinitesimal geometry, three hours. — By Professor O. TEDONE: Maxwell's theory of electromagnetic fields, three hours.

UNIVERSITY OF MESSINA.—By Professor G. BAGNERA: Partial differential equations of the second order, three hours. — By Professor T. BOGGIO: Integral equations with applications to mathematical physics, three hours. — By Professor V. MARTINETTI: Theory of algebraic plane curves and surfaces, curves and surfaces of order 3, three hours.

UNIVERSITY OF NAPLES. — By Professor F. AMODEO: History of mathematics: The XVIII century, three hours. — By Professor A. CAPELLI: Theory of algebraic forms, four and a half hours. — By Professor R. MARCOLONGO: Theory of potential and integral equations, mathematical theory of elasticity, four and a half hours. — By Professor D. MONTESANO: Line geometry, birational transformations of the plane and space, four and a half hours. — By Professor E. PASCAL: Partial differential equations of the second order, three hours. — By Professor L. PINTO: Physical optics, four and a half hours.

UNIVERSITY OF PADUA. — By Professor F. D'ARCAIS: Discontinuous groups of linear transformations, elliptic and modular functions, four and a half hours. — By Professor A. FAVARO: History of optics, especially the invention of the telescope, three hours. — By Professor P. GAZZANIGA: Theory of numbers, three hours. — By Professor T. LEVI-CIVITA: Hydrodynamics, four and a half hours. — By Professor G. RICCI: Mathematical theory of elasticity with applications to optics, four hours. — By Professor F. SEVERI: Theory of continuous groups, two hours; algebraic functions of two variables, two hours. — By Professor G. VERONESE: Synthetic geometry of hyperspaces, four hours.

UNIVERSITY OF PALERMO. — By Professor M. GEBBIA: Mechanics of continua, newtonian attraction, hydrostatics and hydrodynamics, four and a half hours. — By Professor G. B. GUCCIA: General theory of algebraic curves and surfaces, four and a half hours. — By Professor A. VENTURI: Modern views on celestial mechanics, four and a half hours.

UNIVERSITY OF PAVIA. — By Professor E. ALMANSI:

Hydrostatics and hydrodynamics, three hours. — By Professor L. BERZOLARI: Algebraic curves and surfaces, three hours. — By Professor G. VIVANTI: Calculus of variations, three hours.

UNIVERSITY OF PISA. — By Professor E. BERTINI: Geometry of hyperspace, algebraic geometry with applications, three hours. — By Professor L. BIANCHI: Functions of a complex variable, general properties of automorphic functions, four and a half hours. — By Professor U. DINI: Advanced calculus, Bessel functions and integral equations, four and one half hours. — By Professor G. A. MAGGI: Theory of electromagnetic phenomena with special regard to the new hypothesis, four and a half hours. — By Professor P. PIZZETTI: Principles of spherical astronomy, theory of the figures of planets, three hours.

UNIVERSITY OF ROME. — By Professor G. CASTELNUOVO: Algebraic functions of one variable and their integrals, three hours. — By Professor V. CERRUTI: Partial differential equations of the first order, three hours. — By Professor L. ORLANDO: Definite integrals and their application to mathematical physics, three hours. — By Professor V. VOLTERRA: Hydrodynamics, four hours.

UNIVERSITY OF TURIN. — By Professor E. D'OVIDIO: Theory of functions of a complex variable, abelian integrals, three hours. — By Professor G. MORERA: Newtonian potential and attraction of ellipsoids, figure of equilibrium of a rotating fluid mass, three hours. — By Professor C. SEGRE: General survey of concepts and methods in modern geometry, three hours. — By Professor C. SOMIGLIANA: General theory of elasticity, three hours.

PROFESSOR H. POINCARÉ has retired from the professorship of astronomy at the Ecole polytechnique with the title of honorary professor.

AT the University of Toulouse, Professor E. COSSERAT has, at his own request, been transferred to the department of astronomy. Professor J. DRACH, of the University of Poitiers, has been appointed his successor as professor of mathematics.

DR. L. BRICARD, of the Ecole polytechnique, has been appointed professor of mathematics in the conservatory of arts and measures, of Paris, as successor to Professor Lausedat.

DR. G. SANNIA has been appointed docent in geometry at the University of Turin.

DR. G. SCORZA has been appointed docent in projective geometry at the University of Bologna.

PROFESSOR M. ABRAHAM, of the University of Göttingen, has been appointed professor of mathematical physics at the University of Illinois.

DR. A. E. YOUNG, of Purdue University, has been appointed professor of mathematics at Miami College, Oxford, Ohio.

PROFESSOR C. O. GUNTHER, of Stevens Institute, has been promoted to a full professorship of mathematics at the same institution.

MR. E. H. COMSTOCK has been appointed assistant professor of mathematics at the University of Minnesota.

DR. J. H. MCDONALD, of the University of California, has been promoted to an assistant professorship of mathematics.

PROFESSOR O. BOLZA, of the University of Chicago, has been granted leave of absence for the next academic year, which he will spend in Europe.

PROFESSOR J. W. YOUNG, of Princeton University, has been appointed assistant professor of mathematics at the University of Illinois.

AT Princeton University, Drs. C. E. STROMQUIST, J. G. HUN, and C. R. MACINNES have been promoted to assistant professorships of mathematics. Dr. JOHN IRWIN has been appointed instructor in mathematics.

DR. C. N. MOORE has been appointed instructor in mathematics at the University of Cincinnati.

AT the University of Wisconsin, Mr. R. T. CRAIGS has been appointed instructor in mathematics in the university extension department.

MR. W. E. MACDONALD, of the Massachusetts Institute of Technology, has been appointed instructor in mathematics at Harvard University.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

CAMPOS Y CAMPO (J.). Estudio sobre la integracion de las ecuaciones diferenciales lineales. Madrid, 1907. 4to. 79 pp.

CAUCHY (A.). Oeuvres complètes, publiées sous la direction scientifique de l'Académie des Sciences. (2 séries en 27 volumes.) Série I. Volume 2: Mémoires extraits des mémoires de l'Académie des Sciences. Paris, 1908. 4to. Fr. 25.00

CICIONI (G. C.). La divisibilità dei numeri e la teoria delle decimali periodiche. Perugia, Perugina, 1908. 16mo. 11 + 150 pp. L. 2.50

ELFRINK (W. F.). De meetkunde der kegelsneden en eenige harer toepassigen. Harlem, 1907. 8vo. 219 pp. M. 6.50

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Edition française publiée sous la direction de J. Molk. Tome I, volume 3: Théorie des nombres. Rédigé dans l'édition allemande sous la direction de F. Meyer. Fascicule 2. Paris, Gauthier-Villars, 1908. 8vo. Pp. 97–192. Fr. 3.00

ENCYKLOPÄDIE der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Vol. IV (in 4 Teilbänden): Mechanik, redigiert von F. Klein und C. Müller. 1er Teil, 1te Abteilung, 4tes Heft. Leipzig, Teubner, 1908. 8vo. 16 pp. + pp. 435–691. M. 7.80

—— Vol. VI, 2ter Teil: Astronomie, redigiert von K. Schwarzschild. 2tes Heft. Leipzig, Teubner, 1908. 8vo. Pp. 195–334. M. 4.00

FULCO (P.). La derivata logaritmica considerata come un' operazione. Lodi, Wilmant, 1908. 4to. 12 pp.

GARBIERI (G.). Geometria analitica: riassunto di lezioni date nell'università di Genova. Parte I: Luoghi di primo grado, con un cenno su altri luoghi in coordinate cartesiane. Torino, Paravia, 1908. 8vo. 60 pp. L. 3.00

INTERNATIONAL CATALOGUE of scientific literature, published by the Royal Society of London. Mathematics. 6th annual issue (1906). London, 1907. 8vo. 126 pp. 15s.

KLEIN (F.). See ENCYKLOPÄDIE.

KOWALEWSKI (G.). Einführung in die Infinitesimalrechnung, mit einer historischen Uebersicht. Leipzig, 1908. 8vo. 126 pp. M. 1.00

MEYER (F.). See ENCYCLOPÉDIE.

MOLK (J.). See ENCYCLOPÉDIE.

MÜLLER (C.). See ENCYKLOPÄDIE.

NEUBERG (J.). Cours d'algèbre supérieure. Nouvelle édition, augmentée. Liège, 1907. 8vo. 299 pp. Fr. 6.00

OLLERO (D.) Y PÉREZ GRIŠÓN (T.). Curso de cálculo infinitesimal. 2a edición aumentada y revisida. Vol. I: Cálculo diferencial. Vol. II: Cálculo integral. Granada, Ventura, 1907. 14 + 343 + 10 + 328 pp. P. 22.00

PÉREZ GRIŠÓN (T.). See OLLERO (D.).

RUTGERS (J. G.). Einige beschouwingen over de Bessel'sche functies. Alkmaar, 1907. 8vo. 29 pp. M. 1.20

SCHWARZCHILD (K.). See ENCYKLOPÄDIE.

SMITH (R. H.). The calculus for engineers and physicists. 2nd edition, revised and enlarged. London, Griffin, 1908. 8vo. 222 pp. Cloth. 8s. 6d

SYLVESTER (J. J.). Collected mathematical papers. Vol. II: 1854-73, Cambridge, University Press, 1908. 8vo. 747 pp. Cloth. 18s.

VIDAL Y RUA (A.). Aplicaciones geométricas del cálculo integral á la rectificación de líneas, cuadratura de superficies y cubatura de sólidos. 2a edición. Madrid, "Memorial de Ingenieros del Ejercito," 1907. 115 pp. P. 8.00

II. ELEMENTARY MATHEMATICS.

ADLER (A.). Einführung in die Geometrie. Lehr- und Uebungsbuch für die I. Klasse der Realschulen und verwandter Lehranstalten. Wien, Hölder, 1908. 8vo. 3 + 35 pp. M. 0.70

CANNAVIELLO (M.). Corso di geometria elementare per le scuole medie. Parte I: Planimetria. Napoli, Priore, 1908. 8vo. 322 pp. L. 2.00

CRELLE (A. L.). Calculating tables giving the products of every two numbers from 1 to 1000 and their application to the multiplication and division of all numbers above 1000. New edition by O. Seeliger. With tables of square numbers and cube numbers from 1 to 1000. Berlin, Reimer, 1908. 7 + 497 pp. Cloth. M. 15.00

DINO (S. N.). Lezioni di trigonometria, ad uso dei licei d'Italia. 11a edizione. Napoli, Trani, 1908. 8vo. 132 pp. L. 2.25

DORDA Y LÓPEZ HERMOSA (R.). Elementos de cálculo gráfico y monografía y sus aplicaciones prácticas. Texto y atlas. Segovia, Rueda, 1907. 294 pp. P. 12.50

GALLATLY (W.). The nine-point circle. With notes on (1) Simson's line; (2) the radical axis; (3) the quadrilateral. London, Simpkin, 1908. 8vo. 1s.

GOYEN (P.). Advanced arithmetic and elementary algebra and mensuration. London, Macmillan, 1908. 8vo. 3s. 6d.

GREMIGNI (M.). Elementi di geometria, ad uso delle scuole tecniche e professionali. Planimetria, libro III e IV. 3a edizione. Firenze, Bemporad, 1908. 16mo. 146 pp. L. 0.90

HOÜEL (J.). See SCHRÖN (L.).

KAMBLY und LANGGUTH. Arithmetik und Algebra. Nach den preussischen Lehrplänen von 1901 umgearbeitet von A. Thaer. Ausgabe A: Für Gymnasien. 39te Auflage der Kamblyschen Arithmetik und Algebra. Breslau, Hirt, 1908. 8vo. 172 pp. M. 2.00

—. Ausgabe B: Für Oberrealschulen, Realgymnasien und Gymnasien mit mathematischem Reformunterricht. 39te Auflage der Kamblyschen Arithmetik und Algebra. Breslau, Hirt, 1908. 8vo. 248 pp. M. 2.50

LANGGUTH. See KAMBLY.

MACH (E.). Ueber den relativen Bildungswert der philologischen und der mathematisch-naturwissenschaftlichen Unterrichtsfächer. (Schriften des Vereins für Schulreform). Wien, Manz, 1908. 8vo. 32 pp. M. 0.85

MÜLLER (H.). und WITTING (A.). Lehrbuch der Mathematik für die oberen Klassen der höheren Lehranstalten. Leipzig, 1908. 8vo. 12 + 320 pp. M. 3.60

OCTAVIO DE TOLEDO (L.). Tratado de trigonometria rectilínea y esférica. Madrid, Viuda é Hijos de Murillo. 285 pp. P. 8.00

PICATOSTE (F.). Elementos de matematicas. Algebra. Obra declarada de texto para España y Ultramar. Novena edición corregida y aumentada. Madrid, 1908. 208 pp. P. 7.00

ROHRBACH (C.). Vierstellige logarithmisch-trigonometrische Tafeln nebst einigen physikalischen und astronomischen Tafeln, für den Gebrauch an höheren Schulen zusammengestellt. 5te Auflage, Gotha, Thienemann, 1908. 8vo. 36 pp. M. 1.00

SCHRÖN (L.). Tables de logarithmes à sept decimales, pour les nombres depuis 1 jusqu' à 108000, et pour les fonctions trigonométriques de dix en dix secondes. Précédés d'une introduction francaise par J. Hoüel. Edition stéréotype, revue et corrigée. Nouveau tirage. Paris, Gauthier-Villars, 1908. 8vo. 14 + 554 pp. Fr. 10.00

SEELIGER (O.). See CRELLE (A. H.).

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III. APPLIED MATHEMATICS.

BUCKLEY (R. B.). Facts, figures and formulæ for irrigation engineers. London, Spon, 1908. 8vo. Cloth. 10s. 6d.

DESPAUX (A.). Explication mécanique des propriétés de la matière. Cohésion, affinité, gravitation, etc. Paris, Alcan, 1908. 8vo. 365 pp. Fr. 6.00

FORMULARIO di matematica e fisica. 7a edizione aumentata. Firenze, Le Monnier, 1908. 10 + 176 pp. L. 1.00

HENRIQUES (P. H.). Föreläsningar i beskrifvande geometri vid Tekniska Högskolan i Stockholm. Häfte 2. Stockholm, 1907. 8vo. 134 pp. M. 2.40

JAMIESON (A.). Elementary manual on applied mechanics. 8th edition, revised and enlarged. London, Griffin, 1908. 8vo. 484 pp. Cloth. 3s. 6d.

SARRETTE (H.). Précis d'arithmétique des calculs d'emprunts à long terme de valeurs mobilières. Paris, 1908. 8vo. 9 + 287 pp. Fr. 10.00

SCHLENSCHA (J.). See WILDT (J.).

SPOONER (H. J.). Machine design, construction and drawing. A textbook for the use of young engineers. London, Longmans, 1906. 8vo 712 pp. Cloth. 10s. 6d.

WILDT (J.). Praktische Beispiele aus der darstellenden Geometrie. Herausgegeben von J. Schlenscha. Lieferung III. Wien, 1908. 8vo. M. 17.00

SEVENTEENTH ANNUAL LIST OF PAPERS.

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES TO THE PLACES OF THEIR PUBLICATION.

ALLEN, R. B. On Hypercomplex Number Systems Belonging to an Arbitrary Domain of Rationality. Read Feb. 25, 1905. *Transactions of the American Mathematical Society*, vol. 9, No. 2, pp. 203–218; Apr., 1908.

BIRKHOFF, G. D. On the Asymptotic Character of the Solutions of Certain Linear Differential Equations Containing a Parameter. Read (Chicago) Mar. 30, 1907. *Transactions of the American Mathematical Society*, vol. 9, No. 2, pp. 219–231; Apr., 1908.

BLISS, G. A. On the Inverse Problem of the Calculus of Variations. Read (Chicago) Apr. 22, 1905. *Annals of Mathematics*, ser. 2, vol. 9, No. 3, pp. 127–140; Apr., 1908.

—— A New Form of the Simplest Problem of the Calculus of Variations. Read Apr. 27, 1907. *Transactions of the American Mathematical Society*, vol. 8, No. 3, pp. 405–414; July, 1907.

—— A Method of Deriving Euler's Equation in the Calculus of Variations, Read Feb. 29, 1908. *American Mathematical Monthly*, vol. 15, No. 3. pp. 47–54; Mar., 1908.

BOLZA, O. On Lagrange's Method of Multipliers in the Calculus of Variations. Read (Chicago) Dec. 28, 1906. *Mathematische Annalen*, vol. 64, No. 3, pp. 370–387; Sept., 1907.

—— Existence Proof for a Field of Extremals Tangent to a Given Curve. Read (Chicago) Mar. 30, 1907. *Transactions of the American Mathematical Society*, vol. 8, No. 3, pp. 399–404; July, 1907.

BROWN, E. W. On the Lunar Inequality due to the Motion of the Ecliptic and Figure of the Earth. Read Dec. 27, 1907. *Monthly Notices of the Royal Astronomical Society*, vol. 68, No. 6, pp. 450–455: Apr., 1908.

BUCHANAN, H. E. Note on the Convergence of a Sequence of Functions of a Certain Type. Read (Chicago) Dec. 30, 1907. *Annals of Mathematics*, ser. 2, No. 3, vol. 9, pp. 123–126; Apr., 1908.

CAJORI, F. Notes on the History of the Slide Rule. Read (Southwestern Section) Nov. 30, 1907. *American Mathematical Monthly*, vol. 15, No. 1, pp. 1–5; Jan., 1908.

CARMICHAEL, R. D. Multiply Perfect Numbers of Four Different Primes. Read Oct. 27, 1906. *Annals of Mathematics*, ser. 2, vol. 8, No. 4, pp 149–158; July, 1907.

—— On Dividing an Angle into Parts Having the Ratios of Any Given Straight Lines. Read Feb. 23, 1907. *American Mathematical Monthly*, vol. 14, Nos. 6–7, pp. 115–117; June–July, 1907.

—— On Constructing a Cube Having a Given Ratio to a Given Cube. Read (Chicago) Mar. 30, 1907. *American Mathematical Monthly*, vol. 14, No. 10, pp. 174–176; Oct., 1907.

—— Note on Certain Inverse Problems in the Simplex Theory of Numbers. Read Sept. 5, 1907. *Bulletin of the American Mathematical Society*, vol. 14, No. 2, pp. 74–77; Nov., 1907.

—— On Certain Transcendental Functions Defined by a Symbolic Equation. Read Sept. 5, 1907. *American Mathematical Monthly*, vol. 15, No. 4, pp. 78–83; Apr., 1908.

—— On the Classification of Plane Algebraic Curves Possessing Fourfold Symmetry with Respect to a Point. Read Sept. 5, 1907. *Annals of Mathematics*, ser. 2, vol. 9, No. 2, pp. 53–56; Jan., 1908.

—— On a Certain Class of Quartic Curves. Read Oct. 26, 1907. *American Mathematical Monthly*, vol. 15, No. 1, pp. 7–10; Jan., 1908.

CHESSIN, A. S. On an Integral Appearing in Photometry. Read (Southwestern Section) Nov. 30, 1907. *Bulletin of the American Mathematical Society*, vol. 14, No. 5, pp. 212–215; Feb., 1908.

COLPITTS, E. C. On Twisted Quintic Curves. Read Apr. 28, 1906. *American Journal of Mathematics*, vol. 29, pp. 309–344; Oct., 1907.

COOLIDGE, J. L. The Equilong Transformations of Space. Read Oct. 26, 1907. *Transactions of the American Mathematical Society*, vol. 9, No. 2, pp. 178–182; Apr., 1908.

CRATHORNE, A. R. Hilbert's Invariant Integral in the General Isoperimetric Problem. Read (Chicago) Dec. 30, 1907. Included in author's doctor dissertation: Das räumliche isoperimetrische Problem, Göttingen, 1907.

CURTISS, D. R. The Vanishing of the Wronskian and the Problem of Linear Dependence. Read Dec. 28, 1905. *Mathematische Annalen*, vol. 65, No. 2, pp. 282–298; Feb., 1908.

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DICKSON, L. E. Generational Relations for the Abstract Group Simply Isomorphic with the Linear Fractional Group in the $GF[2^n]$. Read (Chicago) Apr. 2, 1904. *Proceedings of the London Mathematical Society*, vol. 35, pp. 443–454; Aug., 1903.

—— The Abstract Form of the Special Linear Homogeneous Group in an Arbitrary Field. Read (San Francisco) Sept. 30, 1905. *Quarterly Journal of Pure and Applied Mathematics*, vol. 38, No. 2, pp. 141–145; Jan., 1907.

—— The Abstract Form of the Abelian Linear Groups. Read Sept. 3, 1906. *Quarterly Journal of Pure and Applied Mathematics*, vol. 38, No. 2, pp. 145–158; Jan., 1907.

—— Representations of the General Symmetric Group as Linear Groups in Finite and Infinite Fields. Read (Chicago) Mar. 30, 1907. *Transactions of the American Mathematical Society*, vol. 9, No. 2, pp. 121–148; Apr., 1908.

—— Modular Theory of Group Matrices. Read Sept. 6, 1907. *Transactions of the American Mathematical Society*, vol. 8, No. 3, pp. 389–398; July, 1907.

—— On Quadratic Forms in a General Field. Read (San Francisco) Sept. 28, 1907. *Bulletin of the American Mathematical Society*, vol. 14, No. 3, pp. 108–115; Dec., 1907.

—— On Triple Algebras and Ternary Cubic Forms. Read Oct. 26, 1907. *Bulletin of the American Mathematical Society*, vol. 14, No. 4, pp. 160–169; Jan., 1908.

—— On Higher Congruences and Modular Invariants. Read Feb. 29, 1908. *Bulletin of the American Mathematical Society*, vol. 14, No. 7, pp. 313–318; Apr., 1908.

—— Criteria for the Irreducibility of a Reciprocal Equation. Read (Chicago) Apr. 17, 1908. *Bulletin of the American Mathematical Society*, vol. 14, No. 9, pp. 426–430; June, 1908.

—— The Galois Group of a Reciprocal Quartic Equation. Read (Chicago) Apr. 17, 1908. *American Mathematical Monthly*, vol. 15, No. 4, pp. 71–78; Apr., 1908.

—— On the Last Theorem of Fermat. Read (Chicago) Apr. 17, 1908. *Messenger of Mathematics*, vol. 38, Nos. 1–2, pp. 14–32; May–June, 1908.

EISENHART, L. P. Surfaces with Isothermal Representation of their Lines of Curvature and their Transformations. Read Apr. 27, 1907. *Transactions of the American Mathematical Society*, vol. 9, No. 2, pp. 148–177; Apr., 1908.

—— Surfaces with the Same Spherical Representation of their Lines of Curvature as Spherical Surfaces. Read Feb. 24, 1906. *American Journal of Mathematics*, vol. 30, No. 1, pp. 19–42; Jan., 1908.

EPSTEEN, S. The Probable Error of a Measurement a Unit in Length. Read (Southwestern Section) Dec. 1, 1906. *University of Colorado Journal of Engineering*, vol. 1, no. 3, pp. 17–18; 1907.

FITE, W. B. Concerning the Degree of an Irreducible Linear Homogeneous Group. Read Sept. 6, 1907. *Bulletin of the American Mathematical Society*, vol. 14, No. 7, pp. 326–329; Apr., 1908.

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—— On the Apsidal Angle in Central Orbits. Read Apr. 27, 1907. *Bulletin of the American Mathematical Society*, vol. 14, No. 1, pp. 6–16; Oct., 1907.

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—— Certain Trajectories Common to Different Laws of Central Force. Read Sept. 6, 1907. *Astronomical Journal*, vol. 26, No. 1, pp. 3–4; Apr. 17, 1908.

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HASEMAN, C. Some Boundary Problems in the Theory of Functions. Read (Chicago) Dec. 30, 1907. Included in the author's doctor dissertation: Anwendung der Theorie der Integralgleichungen auf einige Randwertaufgaben in der Funktionentheorie, Göttingen, 1907.

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KASNER, E. Isothermal Systems in Dynamics. Read Oct. 26, 1907. *Bulletin of the American Mathematical Society*, vol. 14, No. 4, pp. 169–172; Jan., 1908.

—— The Inverse of Meusnier's Theorem. Read Apr. 25, 1908. *Bulletin of the American Mathematical Society*, vol. 14, No. 10, pp. 461–465; July, 1908.

KELLOGG, O. D. Potential Functions on the Boundary of Their Regions of Definition. Read (Southwestern Section) Dec. 1, 1906. *Transactions of the American Mathematical Society*, vol. 9, No. 1, pp. 39–50; Jan., 1908.

—— Double Distributions and the Dirichlet Problem. Read (Southwestern Section) Dec. 1, 1906. *Transactions of the American Mathematical Society*, vol. 9, No. 1, pp. 51–66; Jan., 1908.

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—— Two Theorems in the Geometry of Continuously Turning Curves. Read (Chicago) Apr. 18, 1908. *American Mathematical Monthly*, vol. 15, No. 5, pp. 100–105; May, 1908.

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LAMBERT, P. A. On the Solution of Algebraic Equations in Infinite Series. Read Apr. 25, 1908. *Bulletin of the American Mathematical Society*, vol. 14, No. 10, pp. 467–477; July, 1908.

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—— A Discussion by Synthetic Methods of the Covariant Conic of Two Given Conics. Read (San Francisco) Feb. 29, 1908, *American Mathematical Monthly*, vol. 15, No. 2, pp. 29–30, Feb., 1908.

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LOVETT, E. O. On a Class of Periodic Solutions in the Problem of Four Bodies. Read Dec. 27, 1907. *Annali di Matematica*, ser. 3, vol. 14, No. 4, pp. 327–333; Apr., 1908.

LUNN, A. C. The Deduction of the Electrostatic Equations by the Calculus of Variations. Read (Chicago) Apr. 17, 1908. *Bulletin of the American Mathematical Society*, vol. 14, No. 10, pp. 477–481; July, 1908.

MASON, M. The Expansion of a Function in Terms of Normal Functions. Read Sept. 3, 1906 and Dec. 28, 1906. *Transactions of the American Mathematical Society*, vol. 8, No. 4, pp. 427–432; Oct., 1907.

—— Note on Jacobi's Equation in the Calculus of Variations. Read Feb. 29, 1908. *Bulletin of the American Mathematical Society*, vol. 14, No. 7, pp. 318–321; Apr., 1908.

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—— On the Groups Generated by Two Operators of Order Three Whose Product is of Order Four. Read (San Francisco) Feb. 23, 1907. *Prace Matematyczno-Fizyczne*, vol. 18, pp. 235–240; 1907.

—— Groups Defined by the Orders of Two Generators and the Order of Their Commutator. Read (Chicago) Mar. 30, 1907. *Transactions of the American Mathematical Society*, vol. 9, No. 1, pp. 67–78; Jan., 1908.

—— Groups Generated by n Operators each of Which is the Product of the $n-1$ Remaining Ones. Read Apr, 27, 1907. *American Journal of Mathematics*, vol. 30, No. 1, pp. 93–98. Jan., 1908.

—— Third Report on Recent Progress in the Theory of Groups of Finite Order. Read Sept. 6, 1907. *Bulletin of the American Mathematical Society*, vol. 14, No. 2, pp. 78–91; Nov., 1907, and vol. 14, No. 3, pp. 124–133; Dec., 1907.

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